Kriging in the Presence of LVA Using Dijkstra's Algorithm

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One concern with current geostatistical practice is the use of a stationary variogram to describe spatial structure. Methods have been developed to reduce the reliance on the assumption of stationarity such as: separating data into multiple domains and modeling independently; or using estimation techniques that are less dependent on the assumption of stationarity (ordinary kriging, indicator geostatistics, plurigaussian kriging, and multiple point statistics). In the proposed shortest anisotropic path kriging, locally varying directions and degree of continuity is used to relax the assumption of a stationary variogram. The practitioner will provide the continuity/anisotropy map that contains the directions and degree of continuity in the domain of interest. The best linear unbiased estimator (kriging) is used, but distances track through the continuity map to enforce realistic geological features and non-stationary behavior.

Introduction

The assumption of stationarity is required in order to perform any geostatistical analysis. This amounts to assuming that the model statistics (histogram and variogram) are constant in the modeling domain. Natural phenomenon show complex patterns of spatial variation that may not be represented well by stationary parameters. Common departures from stationarity include: differing directions of continuity; differing behaviours of high and low valued areas; abrupt changes in the variogram across different rock types; or smooth variations in the variogram direction within the modelling domain. Such departures from stationarity are difficult to incorporate into traditional kriging based techniques that rely on a single variogram.

The proposed methodology considers that the variogram has locally varying directions and anisotropy ratios. Such locally varying variograms have been considered in past work by locally orienting a kriging search window and variogram for the estimation location (Deutsch and Lewis, 1992; Xu, 1996; Sullivan et al. 2007). While this touches on the problem of locally varying directions of continuity, there is still an issue if the locally varying variogram is not stationary within the search window, consider the case in Figure 1 where the variogram at the estimation location (gray cell) is different from the variogram of the surrounding area. While this is an extreme case, it serves to illustrate the limitation of applying the local variogram to the entire search neighbourhood in estimation.

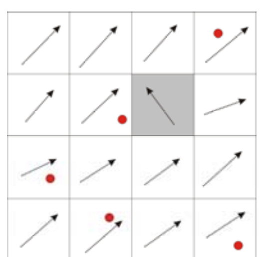
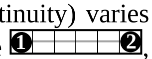


Figure 1: Local variogram directions (arrows) and conditioning data (circles). The variogram at the gray estimation location is inconsistent with the surrounding directions.

Recently two other approaches have been suggested to relax the assumption of a stationary variogram. The first approach is an iterative image analysis technique (Stroet and Snepvangers, 2005) that can reproduce the locally varying directions found in many natural resources deposits as long as there is sufficient data to explicitly control the locally varying directions. The second approach is a 1D spectral simulation (Yao et al., 2007) to honor locally varying directions that has not been tested in three dimensions.

Our proposed technique to integrate locally varying variograms is to use a continuity map (a map of the variogram direction and anisotropy ratios) in kriging. This continuity map is incorporated by modifying the distance between points. When calculating distance, the direction of continuity and the anisotropy ratio (a measure of the degree of continuity) varies within the model; when the path between two points crosses multiple blocks, as in this case , the distance between points 1-2 is calculated as the summation of the four distances in each block where the continuity direction and anisotropy ratio are different. Moreover, the calculation of the distance between points considers non-linear paths. An optimum path between points considering the continuity map is determined using optimization techniques. This optimum path between points is non-linear and is always shorter or equal to the distance using the straight-line path. In this case , the curved path between points 1-2 will be shorter than the straight-line path because of the continuity directions (shown as arrows). In the kriging framework, using this shorter distance results in more realistic covariances between points; this has the effect of generating maps of estimates that can reproduce complex non-linear geometry, such as channels and veins.

The paper will be organized as follows: first, the Methodology section will describe the changes to kriging to incorporate the new distance calculation. Two techniques will be applied to generate the optimal path (i) a Newton method optimization and (ii) a solution based on Dijkstra's algorithm using graph theory. The Results and Discussion section will discuss a short 2D case study. Although the technique works equally well in 3D, space requirements demand a 2D case study. This will be followed by a Conclusions section which contains a discussion of future work.

Methodology

The proposed methodology uses kriging with modified distances / covariances between points. Readers will find an easy-to-understand discussion of kriging in Isaaks and Srivastava (1989) and implementation details for the program KT3D in Deutsch and Journel (1998). The significant change to the methodology of KT3D is to utilize the optimized path in the presence of locally varying directions, specifically the methodology to estimate a kriging mean and variance at an unknown location is:

1) Search for nearby data using a user defined search ellipsoid.
2) Calculate the covariance between all nearby data and the estimation location. This is done using the optimized distance and a variogram provided by the user.
3) Solve the kriging system of equations to generate weights for the nearby data.
4) Determine the kriging mean and variance.

Only step 2 above is different than a traditional implementation of kriging with a search ellipsoid. The generation of the optimal path (and thus distance) between two points will be expanded upon below. For all other steps of the kriging process the reader is referred to KT3D (Deutsch and Journel, 1998).

Optimum Path

Consider the example in Figure 2, where the underlying anisotropy field could be characteristic of a channellized petroleum reservoir or a curvilinear mineralized zone. The distance between points 1 and 2 should be calculated along the direction of the deposit following the gray arrows. Unfolding could be used to incorporate the need to follow along the deposit; however, unfolding is a complicated task with a significant degree of subjectivity, particularly when there are many local variations in the directions of continuity. Rather than unfold the deposit, the shortest optimal path through the deposit could be calculated.

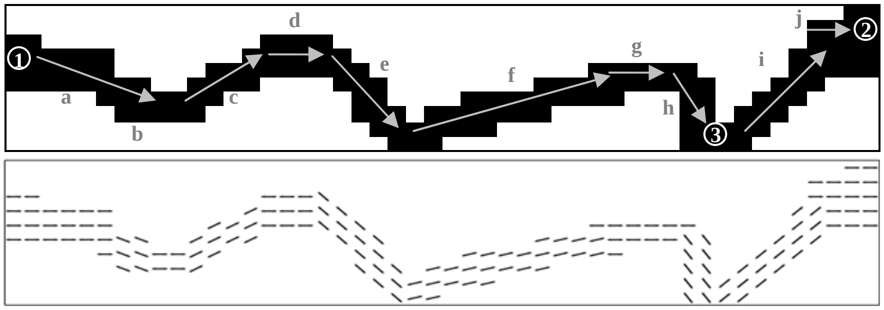


Figure 2: Above: LVA field example. It would not be appropriate to consider the straight-line path between points 1 and 2. More appropriately, the curvilinear path following the arrows could be used. Below: Typically the directions are specified in all areas of the model.

With an appropriate locally varying directional field (i.e. following the deposit as in Figure 2) and an appropriate locally varying anisotropy ratio field (i.e. perhaps a high ratio in the deposit and a low ratio outside) the optimal path between points 1 and 2 will be through the channel. In the context of this paper, we will consider the optimal path such that the anisotropic distance (Equation 1) between two points p_1 and p_2 is minimized. The distance is calculated for each different anisotropic zone (i.e. sum distances *a* through *j* in Figure 2). This will result in the maximum covariance between two points using a typical covariance function.

$$d_{ij}^{\,2}(\mathbf{p}_1,\mathbf{p}_2) = (\mathbf{p}_2 - \mathbf{p}_1)^T R^T R(\mathbf{p}_2 - \mathbf{p}_1) \tag{1}$$

The rotation matrix is defined by anisotropy ratios (r_1 and r_2) and angles ($\alpha\ \beta\ \gamma$):

$$R = \begin{bmatrix} \cos\beta\cos\alpha & \cos\beta\sin\alpha & -\sin\beta \\ \frac{1}{r^1}(-\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha) & \frac{1}{r^1}(\cos\gamma\cos\alpha + \sin\gamma\sin\beta\sin\alpha) & \frac{1}{r^1}(\sin\gamma\cos\beta) \\ \frac{1}{r^2}(\sin\gamma\sin\alpha + \cos\gamma\cos\beta\cos\alpha) & \frac{1}{r^2}(-\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha) & \frac{1}{r^2}(\cos\gamma\cos\beta) \end{bmatrix} \tag{2}$$

Two techniques will be used to determine the optimal path; first, the use of graph theory will be applied to the problem. In a graph all nodes (grid cell centers) are connected to adjacent nodes via links. These links are predetermined by the user and assigned a distance value. This distance represents the anisotropic distance between two nodes (or two grid cells). A possible graph where all cells are linked to their neighbours is shown in Figure 3. There is no theoretical limitation to the number of links between nodes; however, each link added to the graph increases the CPU time required to determine the minimum distance between points. The distance between points must follow the links as in Figure 3.

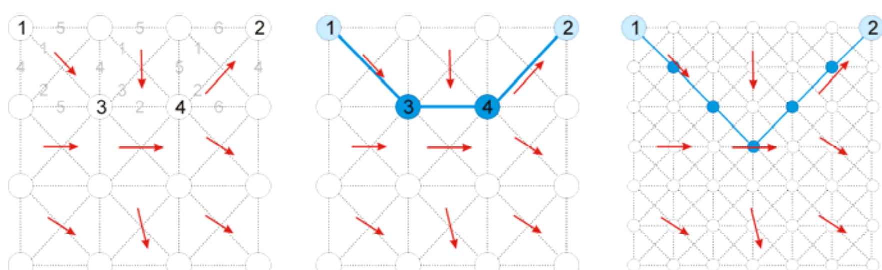


Figure 3: Left - Graph representation of a 2D grid where each grid center is connected to adjacent grid centers, relevant distances are indicated on the graph. Middle - Potential path between nodes 1 and 2. Right - The optimal path may be curved. A limitation of the graph is that the path must go though nodes, refining the graph allows for more flexibility in the shape of the path.

If the problem is formulated in this way, with all permissible paths restricted to the links, then there are algorithms for solving for the path that results in the minimum distance between any two points: the

Dijkstra algorithm (Dijkstra, 1959); the Bellman-Ford algorithm (Cormen et al., 2003); and the A* search algorithm (Hart et al., 1968) to name a few. In this case the Dijkstra algorithm was implemented. All algorithms will determine the optimal path; however, there is a difference in the speed of the different algorithms. Issues of CPU time will be deferred to the discussion section. The interested reader is referred to Dijkstra (1959) for a description of the implemented algorithm. Figure 4 shows a small test example with the optimal distances and paths shown. The restriction of the path to following nodes is apparent.

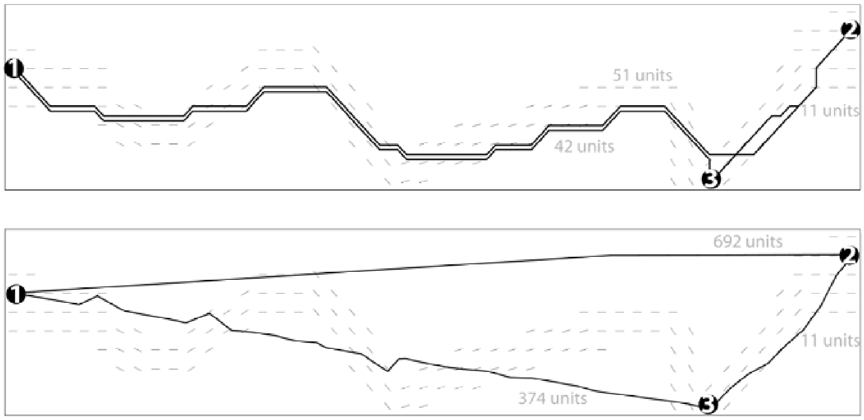


Figure 4: Example optimum paths using the LVA field in Figure 2. Above – Using the Dijkstra algorithm the global solution is discovered but the path is restricted in shape. Below – Using the Newton method (described below) to determine the optimum path; the global solution is not discovered but the path is unrestricted in shape

An alternative to using graph theory to determine the optimal path is to consider the problem an optimization problem (a more extensive review of this method can be found in Boisvert et al. (2007)). The premise is to optimize the path between points by inserting control nodes on the straight-line path and adjusting the nodes such that the resulting path is shorter than the initial path (Figure 5). The resulting nonlinear path will be shorter because of the non-stationary anisotropy.

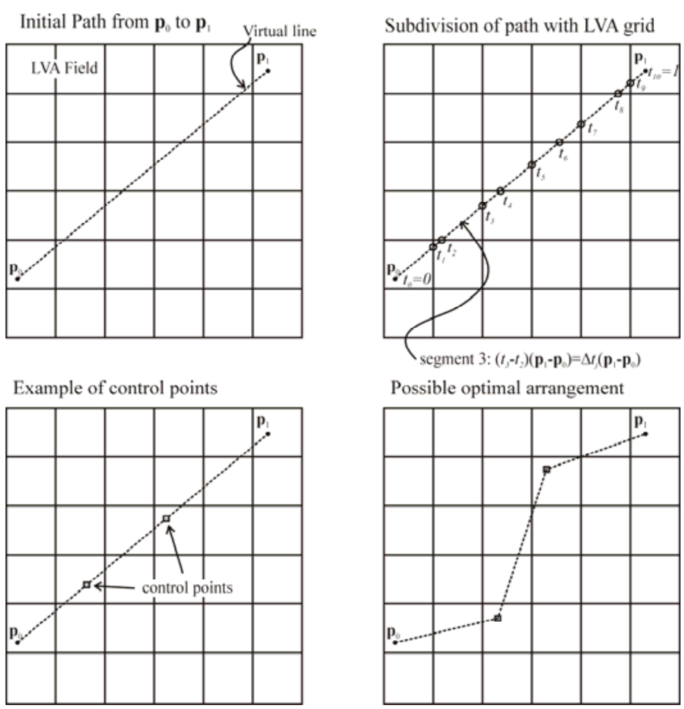


Figure 5: Steps in minimizing the distance between two points with the Newton method optimization. (Source: Boisvert et al. 2007)

Newton method optimization (Boyd and Vandenberghe, 2004) is used to determine the direction to move the control nodes such that the distance is minimized. A line search is then used to determine the magnitude of the move in the aforementioned direction.

The general description of the technique is that it uses the first and second derivatives, the Joacobian and Hessian respectively, to determine the direction of the minimum path. An analogy to a 2D function is that the first derivative (the gradient) always points towards the decreasing direction of a function. In this case the function is the distance and the Jacobian (first derivative) will point towards the direction of minimizing the distance. This direction is the direction in which to move the nodes (Figure 5). Once this direction is determined the golden section search determines the optimal distance to move the control points.

Readers interested in the details of the Newton method optimization or the golden section search are refereed to Boyd and Vandenberghe (2004) for a general discussion or Boisvert et. al (2007) for a discussion of the technique as implemented in this paper.

There are some differences between the paths discovered by the two methods presented. Newton method optimization results in a local minimum and may not find the global minimum path between nodes (Figure 4). Conversely, the graph theory implementation is guaranteed to determine the minimum distance between nodes but with the restriction imparted by the links between nodes. Paths that may result in shorter distances may not be discovered in the graph because of the limitation of linking nodes.

Generation of Covariance from Optimized Distance

At this point the distance between any two points in a locally varying angle/anisotropy ratio field can be determined; however, in the kriging framework the necessary input parameter is the covariance between points. To determine this covariance the user must define a covariance function to calculate the covariance from the optimized distance. This covariance function is assumed stationary across the deposit. This is not the same as assuming a stationary variogram as is traditionally done in kriging because the optimal distance depends on the underlying non-stationary directions and anisotropy ratios. It would be conceivable to use a different covariance function at all locations in the model; however, there is rarely sufficient data to define such a nonstationary covariance function.

Results and Discussion – A Case Study

The results and discussion of the proposed methodology will center around a case study. Recall that the goal of this strategy is to generate nonlinear geological features, such as curvilinear channels or multiple striking vein sets with varying azimuths. Such features are typically difficult (if not impossible) to generate with kriging unless there is sufficient data to control the features.

The case study begins with an appropriate LVA field. The generation of this field has receive little attention in this paper and is of great importance. This field will control the geological characteristics of the resulting estimation maps. It is the duty of the practitioner to generate an appropriate LVA field using all the available knowledge (geological interpretations, secondary data, expert opinion, etc.). Often there will be uncertainty in the LVA field; in such cases multiple LVA fields should be considered and multiple estimation maps generated spanning the uncertainty in the LVA field. In this case study two geological interpretations will be analyzed; a stratified folded deposit and a channel/vein like interpretation. Consider the two LVA fields in Figure 6.

The underlying geology in both of these cases is apparent in the resulting kriging maps. The complex geology is not apparent in the data but can be incorporated using the underlying LVA field. The importance of the LVA field must be stressed. If there is uncertainty in the LVA field it should be accounted for by considering multiple scenarios, such as the folded deposit and vein deposit scenarios presented in Figure 6. This is analogous to considering multiple variograms in a traditional kriging framework when there is uncertainty in the variogram.

Considering run time, the graph theory implementation is quadratic (or worse) with the number of nodes and links between nodes; large models are not possible with the current naïve implementation. Future work will be to use clever programming structures (for example, see Ahuja et al. 1990, Thorup 1999 or Raman 1997) to improve run time. Moreover, restricting the graph to the search radius of the local kriging systems will significantly reduce CPU time. Currently, both the Dijkstra and the Newton method are not feasible with large models; for the case study shown in Figure 6 the Newton method takes 20.4 minutes and Dijkstra's algorithm 16.6 minutes using 8 data in each kriging system.

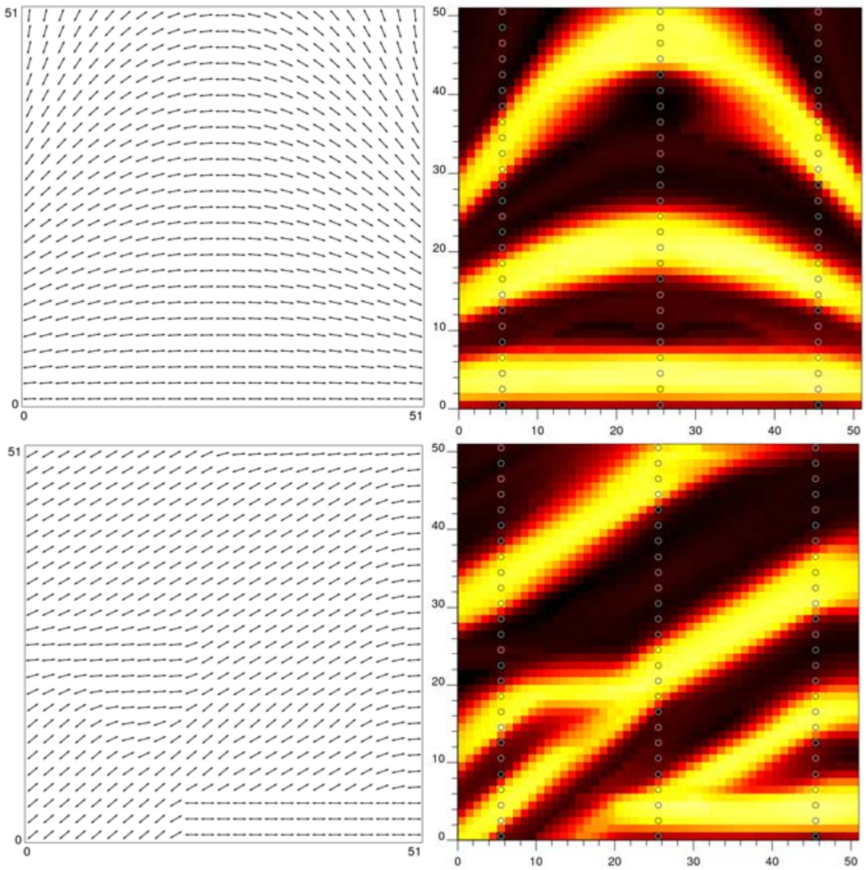


Figure 6: Above Left: A folded seam deposit. Above Right: kriging with the folded LVA considering the graph methodology presented. Below Left: a channel/vein deposit. Below Right: kriging with vein LVA considering the graph methodology presented. The conditioning data are overlain with three drillholes and 78 total samples.

Conclusions

Although the methodology presented can be used to generate realistic geological maps, there are a number of drawbacks and limitations in its present implementation. First, because the straight line path is not used, there is no guarantee of a positive definite system of kriging equations. The authors experience with the method indicates that there are very few cases when these systems are indefinite (for reasonable anisotropy ratios and variogram ranges less than the field size). Moreover, using the graph theory implementation typically results in fewer indefinite matrices than the Newton method because of more constant paths and the determination of the global solution. Future work in this area should be directed at generating a stable system of equations where positive definiteness is guaranteed.

The possibility of indefinite matrices occurring is off-putting for some practitioners; however, it has been found that they seldom occur, and if they do occur they are very easily identified with common mathematical tests and fixed with common mathematical tools (see Saad and Vorst (2000) for an extensive review of solving indefinite matrices). There should be more attention focused on determining if the adjustment of the distances to follow the nonlinear paths is warranted or not. It may be more erroneous to ignore departures from stationarity, as in Figure 6, by using a single variogram than to be overly concerned with the possibility of generating indefinite matrices.

The second limitation of the proposed methodology is the CPU intensive path optimization. Because of the number of paths that must be considered in a typical kriging run, the optimization of the path must be extremely fast. As discussed in the case study, future work will be directed at improving the speed of the optimization.

As with any geostatistical tool, its implementation should be considered based on its advantages (generating non-stationary estimates) and disadvantages (possibility of indefinite matrices). With future research the authors are hoping to formulate a methodology that will retain the stability of traditional kriging (no indefinite matrices) and consider LVA. It is hoped that the presentation of this technique will encourage other researches to develop methodologies to incorporate LVA into geostatistical modeling and to help reduce the dependency of geostatistics on the assumption of stationarity.

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