Conditional Bias of Geostatistical Simulation for Estimation of Recoverable Reserves

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Abstract

Conditional bias is an infamous problem with estimation methods including kriging. Changing estimation parameters will mitigate, but not remove, conditional bias. The conditional bias of kriging is well understood; however, there is widespread confusion in the literature and among practicing geostatisticians regarding the conditional bias of geostatistical simulation. There is no conditional bias of simulation when the simulation results are used correctly. The correct use of simulation for recoverable reserve estimation is to (1) generate multiple realizations conditional to all available data at a small scale, (2) linearly average all realizations to the chosen block size and (3) calculate the probability of each block being ore and the ore grade of each block. The "probability of ore" and the "ore grade" are conditionally non-biased.

Introduction

The problems of conditional bias are notorious. Although there have been significant developments in geostatistics, conditional bias is still a concern of practical ore reserve estimation. Most mines base recoverable reserves on some type of estimate. Such estimates are often conditionally biased. Attempts to improve estimation algorithms have met with some success, but the problems of conditional bias persist.

Conditional bias occurs when the expected value of the true grade (Z_V) conditional on the estimated grade ($Z^*_V = z$) is not equal to the estimated grade, that is:

$$E\{Z_V \mid Z^*_V = z\} \neq z \tag{1}$$

where the V is symbolic of some volume of estimation, for example, a selective mining unit (SMU) volume. Conditional non-bias occurs when the expected value of the true grade conditional on the estimated grade is equal to the estimated grade.

Conditional bias is almost always present due to the smoothing effect of kriging in the presence of sample data that are relatively widely spaced. The true grade is typically less than the estimated grade when the estimated grade is high and the true grade is typically greater than the estimated grade when the estimated grade is low. Conditional bias is predicted by theory and confirmed by cross validation, see Figure 1. The estimated grade is plotted on the abscissa axis because it is the independent variable (known) and the true grade on the ordinate axis because it is the dependent variable (unknown). The true grades have a greater variance than the estimated grades due to the smoothing effect of kriging. The curved line shows conditional bias in the form of a conditional expectation curve, $E\{Z_V \mid Z^*_V = z\}$.

A small example has been constructed to illustrate local conditional bias. Consider a multivariate Gaussian mineralization with a lognormal histogram with a mean of 1.0 and a variance of 4.0. The variogram has a relative nugget effect of 20% and an isotopic range of 40 m. Consider four samples on a regular 20 m grid to estimate a central 10 m square block. This is a favorable estimation scheme since the nugget effect is relatively low and the range is quite large relative to the sample spacing. Nevertheless, there will be conditional bias. Local high and low-grade cases are shown in Figure 2. The kriged grade is too high in the high-grade case and too low in the low-grade case, which is consistent with the schematic Figure 1. Each sample is assigned the same kriging weight, therefore, the high and low-grade samples, respectively, influence too large an area causing the conditional bias.

There is no question that judicious selection of geological rock types and treatment of trends will mitigate the problems of conditional bias. These decisions must be considered prior to estimation. Most orebodies permit some deterministic modeling of geological controls. Geostatistical estimation and simulation must consider all such interpretive geological information. This should not be forgotten in the following presentation where most of the examples are synthetic with no geological rock type control.

Classically, mine planning is based on block estimates created from some form of kriging. Adequate follow-up with validation exercises on these estimates is important. In practice, however, kriging is implemented with widely spaced sample data and limited access to the true grades at the same support. This often results in smoothed and/or conditionally biased selective mining unit (SMU) estimates and, therefore, often leads to misleading SMU selection, recoverable reserve estimates and profit profiles. There are many flavors of kriging; Uniform Conditioning (UC), Direct Conditioning (DC), Disjunctive Kriging (DK), the MultiGuassian approach (MG), the Bi-Guassian approach (BG) and Median Indicator Kriging (MIK) are a few, see Guibal (1984) and Remacre and Marcotte (1985) and David.

The classical application of kriging to mine planning is discussed in David (1977) and Journel and Huijbregts (1978). The relationship between conditional biases, smoothing and various search routines is well understood and documented. The papers by Krige (1999d, 2000) and Assibey-Bonsu, Krige (1994a, 1996, 1999b, 2000) and Isaaks (1999) and Davis are more recent references.

There are two schools of thought related to the conditional bias and smoothing of ordinary kriging for mine planning:

1. The "conditional bias of block estimates is always wrong" school championed by D.G. Krige, W. Assibey-Bonsu and coworkers. Here, one never accepts block estimates known to be wrong in expected value. Large search routines retaining many conditioning data are implemented to minimize uncertainty and conditional bias. The price, however, is block estimates that are smooth and near the mean.

2. The "let's get recoverable reserves right" school championed by various other practitioners. The idea here is to anticipate the dispersion variance of the true block grades. Fewer samples are used in the kriging plan to increase the variability of the block estimates in the hope of reproducing the true block grade dispersion variance. The price of this approach, however, is block estimates that are conditionally biased.

We see the relative merits of both schools; however, there is no reason to pay with too smooth estimates or conditional bias. We would rather pay with the increased computational and professional time to implement simulation correctly. Creating "probability of ore" and "ore grade" SMU estimates derived from multiple geostatistical simulations eliminates conditional bias, corrects smoothing, and accounts for uncertainty. This paper presents how valid recoverable reserve estimates can be derived from geostatistical simulation.

The essential idea of using geostatistical simulation for the estimation of recoverable reserves is to: (1) apply geostatistical simulation to quantify the uncertainty in block grades and account for smoothing, (2) calculate the "probability of ore" ($\mathbf{P}_{ORE}$) for every SMU as the proportion of simulated block grades above cutoff, and (3) calculate the "ore grade" ($\mathbf{Z}_{ORE}$) for every SMU as the average of the simulated block grades above cutoff.

There are other incorrect ways of using multiple realizations for estimation of recoverable reserves. Simply accepting one realization would be misleading. It would not reflect the range of uncertainty and may show "simulated features" that are unrealistic without a full suite of realizations. The average of several realizations would also be inappropriate since these averages approach kriging. The detriments of accepting one realization or the average of several realizations have been well documented, see Krige (2000) and Assibey-Bonsu and Krige (2001).

Calculating probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ estimates for all the mining blocks allows for improved recoverable reserves for mine planning; however, these estimates must be properly validated and shown to be conditionally non-biased for various cutoff grades. The entire procedure can be largely automated so that the mining engineer/geologist gets maps of the $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ SMU estimates and their corresponding cross validation plots. The $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ block estimates are shown to be conditionally non-biased.

There are numerous advantages to basing mine plans and recoverable reserves on probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates. Such estimates are not conditionally biased, have the right dispersion variance and account for inherent block grade uncertainties. The uncertainty involved in selection is conveniently represented by $\mathbf{P}_{ORE}$ maps and the expected grade of ore within any subsection of the orebody is straightforward to calculate with $\mathbf{Z}_{ORE}$ maps. The increased CPU demand of this approach is not an issue.

The recommended methodology is presented with all necessary detail. A simple example will illustrate the procedure.

Methodology

The language of probability is a well-established way to express uncertainty. Multiple geostatistical realizations of the grades at a fine scale, conditioned by all available data, are constructed with Guassian or indicator simulation techniques to express the uncertainty in the grades:

$$\{z^{(l)}(\mathbf{u}_q),\ q=1,\ldots,F,\ l=1,\ldots,L\} \tag{2}$$

The locations $\mathbf{u}_q$, $q = 1,\ldots, F$ represent a finely gridded version of the orebody. There should be 10 or more "fine" blocks per SMU of volume V and the number of realizations $l = 1,\ldots, L$ should be sufficient to reflect uncertainty and avoid decision making based on unrepresentative stochastic features (20-100 are considered sufficient).

These small-scale realizations are linearly block averaged to the mining size V to obtain a new set of L realizations at the SMU resolution:

$$\{z_V^{(l)}(\mathbf{u}_j),\ j=1,\ldots,N,\ l=1,\ldots,L\} \tag{3}$$

The probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ for the N SMU locations are calculated from the L realizations while applying the cutoff grade z_C. An indicator transform is defined for every mining block, for every realization:

$$\mathbf{i}(z_V{}^{(l)}(\mathbf{u}_j); z_C) = \begin{cases} 1, & if \quad z_V{}^{(l)}(\mathbf{u}_j) > z_C \\ 0, & otherwise \end{cases} \qquad for \quad j = 1,..., N \,;\, l = 1,..., L \tag{4}$$

The probability of ore at each location is calculated:

$$\mathbf{P}_{ORE}(\mathbf{u}_j) = \frac{1}{L}\sum_{l=1}^{L} \mathbf{i}(z_V{}^{(l)}(\mathbf{u}_j); z_C) \qquad for \quad j = 1,..., N; l = 1,..., L \tag{5}$$

The ore grade at each location is calculated:

$$\mathbf{Z}_{ORE}(\mathbf{u}_j) = \frac{\sum_{l=1}^{L} \mathbf{i}(z_V{}^{(l)}(\mathbf{u}_j); z_C) \cdot z_V{}^{(l)}(\mathbf{u}_j)}{\sum_{l=1}^{L} \mathbf{i}(z_V{}^{(l)}(\mathbf{u}_j); z_C)} \qquad for \quad j = 1,..., N; l = 1,..., L \tag{6}$$

The average grade is also calculated as a means to check the kriging estimates:

$$\overline{\mathbf{Z}}(\mathbf{u}_j) = \frac{1}{L}\sum_{l=1}^{L} z_V{}^{(l)}(\mathbf{u}_j) \qquad for \quad j = 1,..., N; l = 1,..., L \tag{7}$$

The probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates are useful for mine planning. The ore grade $\mathbf{Z}_{ORE}$, ore tonnes $\mathbf{T}_{ORE}$ and waste tonnes $\mathbf{T}_{WASTE}$ for a particular subset of N' mining blocks for a particular stage of mining is calculated:

$$\begin{aligned} \mathbf{Z}_{ORE} &= \frac{\sum_{j=1}^{N'} \mathbf{P}_{ORE}(\mathbf{u}_j) \cdot \mathbf{Z}_{ORE}(\mathbf{u}_j)}{\sum_{j=1}^{N'} \mathbf{P}_{ORE}(\mathbf{u}_j)} \qquad for \quad j = 1,..., N' \\ \mathbf{T}_{ORE} &= T_V \cdot \sum_{j=1}^{N'} \mathbf{P}_{ORE}(\mathbf{u}_j) \qquad for \quad j = 1,..., N' \\ \mathbf{T}_{WASTE} &= T_V \cdot \sum_{j=1}^{N'} [1 - \mathbf{P}_{ORE}(\mathbf{u}_j)] \qquad for \quad j = 1,..., N' \end{aligned} \tag{8}$$

where T_V is the tonnes of a full block V. Of course, the specific gravity could be modeled geostatistically on a by-rock-type basis.

These estimates implicitly assume perfect selection at the time of mining, that is, adequate grade control practices to minimize misclassification, see Richmond (2001). Geostatistical simulation could also be used to evaluate the performance of different grade control schemes, but that is not the subject of this paper, see Deutsch, Magri and Norrena (2000).

We must now demonstrate that the $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ SMU estimates are conditionally unbiased, that is, we must show that:

$$
\begin{aligned}
&E\{\mathbf{P}_{ORE}, z_C \mid \mathbf{P}^*_{ORE}, z_C = p\} = p \qquad \forall\ p \in [0,1] \\
&E\{\mathbf{Z}_{ORE}, z_C \mid \mathbf{Z}^*_{ORE}, z_C = z\} = z \qquad \forall\ z
\end{aligned} \tag{9}
$$

These cannot be proven in all generality since the true distribution of grades will never exactly follow our random function (RF) models; however, we could indeed prove these conditionally non-bias conditions if the true grades and the simulated grades follow the same RF model, that is, the uncertainty in the true grades is modeled correctly by the conditional distribution functions $F^*_V(\mathbf{u}_j; z_C)$ sampled by the simulation. In practice, if our simulated realizations reflect the true grades and we have sufficient data to estimate all parameters (variogram, histogram, etc) with no error then $F_V(\mathbf{u}_j; z_C) \approx F^*_V(\mathbf{u}_j; z_C)$ for all locations $\mathbf{u}_j$, all volumes V and all cutoff values z_C. And truncated statistics such as $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ are the same if the distributions are the same.

It is important to note that kriging estimates are conditionally biased even if the correct RF parameters are used. A single realization or the average of several realizations will also be conditionally biased even if the correct RF parameters are used.

The conditional non-bias of $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ estimates can also be demonstrated with extensive data or by a simulation exercise. We could check the true SMU grades collocated with a subset of SMUs N with predicted probability of ore p +/- Δp. There should be $\frac{n}{N} \approx p$ of that subset actually ore, where n is the number of true SMU grades above cutoff. For example, if there are 250 SMUs with predicted probability of ore 0.10 +/- 0.025 then 25 of those SMUs should truly be above cutoff, that is, $\frac{n \ = 25}{N \ = 250} \approx p = 0.10$. And the true ore grade should be approximately equal to the estimated ore grade $\mathbf{Z}_{ORE}$ for every SMU.

The preceding numerical approach is simple. The $\mathbf{P}_{ORE}$ estimates are simply the proportions of grade realizations above cutoff, and the averages of those grades identified to be above cutoff are the $\mathbf{Z}_{ORE}$ estimates. Checking the conditional non-bias of these estimates is simple and is also consistent with the theory of cross validation.

Proper validation of conditionally non-biased estimates is difficult because the true grades at the mining support V are rarely available. Monte Carlo simulation could be used as a *numerical laboratory* for creating $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ estimates and ensuring their conditional non-bias. Although the inference and uncertainty of variogram, histogram and estimation parameters is not covered in this paper, we acknowledge that they are critical issues in practice. The steps for a numerical validation to what has been presented could consist of:

- Building a fine-scale true-grade model ($z(\mathbf{u}_q)$, $q = 1,...,F$).
- Creating an SMU true-grade model ($z_V(\mathbf{u}_j)$, $j = 1,...,N$) by averaging the fine-scale truth.
- Sampling the fine scale truth model at some realistic spacing for exploration data.
- As an aside, kriging the block grades using different searching routines to observe the resulting degrees of conditional bias.

- Simulating multiple geostatistical realizations of the grades at a small scale, conditioned by the exploration data.
- Block averaging the realizations to the mining support *V*.
- Calculating the probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates for a chosen cutoff z_C.
- Validating the $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ estimates to ensure conditional non-bias:
  - Define *K* symmetric probability intervals P_k +/- $\Delta P,\ k = 1,\ldots,K$.
  - Count the number of SMU locations N_K with $\mathbf{P}_{ORE}$ estimates falling within each of the *K* probability intervals.
  - Count the number of true SMU grades n_K that are above cutoff for each subset of N_P collocated SMUs.
  - Compare the proportion of true SMU grades above cutoff $\frac{n_K}{N_K}$ with the center of each of the *K* corresponding probability intervals.
  - Compare the true ore grade to the estimated ore grade $\mathbf{Z}_{ORE}$ for each SMU.
- Repeating the last two steps with different cutoff grades to explore the sensitivity of conditional bias to cutoff grade.

Although well established in science, there are a number of concerns with such a Monte Carlo procedure: (1) the truth models are often too simplistic, and (2) any method that makes use of the underlying random function model used to generate the truth model could appear unrealistically good. Awareness of these concerns has guided the implementation of our methodology below; for example, kriging and simulation are given the same exploration data to make fair comparisons.

We have presented a methodology and accompanying numerical laboratory to create global and local conditionally unbiased $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ SMU estimates. The $\mathbf{P}_{ORE}$ estimates are classified into *K* symmetric probability intervals and validated one probability interval (one subset of SMUs) at a time, while the estimated ore grades $\mathbf{Z}_{ORE}$ are validated one SMU location at a time. We can explore the conditional bias of various estimates derived from various estimation schemes and algorithms for any numerical/geometric subset of SMUs. It is important to realize, however, that it is the sum of localized conditional biases (see Figure 2 for local high and low-grade locations) which gives rise to global conditional biases (see Figure 1).

An Example

The grades in this example are synthetic, but were chosen to mimic practical mineralization. Application to real data is straightforward; however, access to sufficient true grades is limited. The proposed approach and numerical laboratory will work for any particular grade or variable *z*. The parameters used in Krige (2000) and Assibey-Bonsu are used for easy comparison.

The exhaustive true grade model is created by unconditional simulation using the “sgsim” program from GSLIB. A grid of 120 by 120 by 120 (1,728,000) values spaced 1m apart with a lognormal distribution with a mean grade of 1.0 and a variance of 1.7 is built. The variogram has a 41% relative nugget effect with ranges of 19m in the horizontal and 35m in the vertical. The true grade distribution is visualized in Figure 3. The spatial structure is seen on the cross-sections and on the variogram and the lognormal histogram parameters are successfully reproduced as shown by the histogram of true grades.

Two sample sets are now extracted: a 5m grid of samples (13,824 values) that will be used to create a reference grade model at the chosen SMU support and a 20m grid of samples (216 values) that will be used as exploration data for subsequent kriging and simulation. Figure 4 shows the histogram, calculated variogram and model variogram for both sample data sets. Compare to the reference distribution of grades in Figure 3.

Checking the conditional bias of probability of ore and ore grade estimates requires knowing the true grades at the chosen SMU support. An SMU size V of 10 by 10 by 10m is chosen and two true grade models are calculated: the first by ordinary block kriging using the 5m spaced sample data set and the second by linearly block averaging the fine-scale true grade model. Figure 5 shows the results of the two different methods. Compare the two true-grade SMU central XY cross-sections in this figure to the true grade fine-scale central XY cross-section in Figure 3. Also, note the excellent correlation between the two methods as shown by the excellent correlation in the scatterplot. The kriging estimates are more variable than the re-blocked values (0.71 vs. 0.52) since kriging tends to overestimate in high-grade areas and underestimate in low-grade areas, whereas block averaging will tend to average highs and lows out. The first method was considered because of the paper by Krige (2000) and Assibey-Bonsu. We will continue with the true SMU grades created from block averaging.

Ordinary kriging with 4 and 24 maximum data is performed with the "kt3d" program from GSLIB. With 4 data, the true underlying block variance is well reproduced (0.55 obtained vs. 0.52 sought); however, kriging using 24 maximum data causes the variance to be severely underestimated (0.20 vs. 0.52 sought). The distribution of estimated grades from both kriging plans are shown in Figure 6 (compare with the block averaged true grades in Figure 5 and the fine-scale true grade distribution in Figure 3). Figure 7 shows the conditional bias of the kriging results. The slope of the regression line using 24 maximum data is 0.88; with 4 data the slope decreases to 0.53. As expected, the conditional bias when using 4 maximum data is much more severe than when using 24 maximum data.

One hundred fine-scale geostatistical realizations, conditioned by the 20m grid of exploration data are simulated. These realizations are then block averaged to the SMU scale. Figure 8 illustrates this process for the 100th simulation. Notice that the cross-sections and histograms are similar to the fine-scale true-grades (see Figure 3) and the block averaged true SMU grades (see Figure 5) – the mean is slightly lower and the variance is slightly higher because the exploration data were used as input for simulation.

The results of the probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates are summarized in Figure 9. We assumed a base case cutoff grade of 0.5. The high-grade areas have high probabilities of being ore and high expected ore grades. Also, all the expected ore grades are higher than the 0.5 cutoff and the minimum value of the probabilities is 0.17.

To show that the probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates are conditionally unbiased, six symmetric probability intervals were chosen. The centers of these probability intervals are the estimated probabilities. The proportions of true SMU grades above cutoff within each probability interval are the true probabilities. The average grade of ore is calculated. In Figure 10, a cross plot of the true probabilities of ore versus the estimated probabilities of ore and a cross plot of the true ore grades versus the estimated ore grades is shown. Notice the perfect correlation between the true and estimated probabilities of ore (the first dot was omitted because there was no probability of ore estimate within this symmetric probability interval) and the agreement between the mean of true ore and estimated ore grades (the estimated mean is slightly lower than the true mean because the exploration data was input into "sgsim").

We explored cutoff grades at 0.5, 1.0, and 1.5 (below, at and above the mean). The results are shown in Figure 11. In all cases, perfect correlation between the true and estimated probabilities of ore exists. The same sensitivity analysis is then done for the ore grade estimates. These results are given in Figure 12 in the form of a table. The agreement of true and estimated ore grades also persists.

We now investigate conditional bias at two individual SMU locations found on the central XY cross-section, see Figure 13. The SMU locations have $\mathbf{P}_{ORE}$ estimates of 1.00 and 0.28, true grades of 2.70 and 0.29, mean grades of 2.50 and 0.45 and fall into the 6th and 2nd probability intervals, respectively. The true grade, the distribution of simulated grades and ordinary kriging estimates with 4 and 24 maximum conditioning data are compared at each location in this figure (the mean of the distributions are different from the ordinary kriging estimates since simple kriging is used in conditional simulation in order to reproduce the spatial structure). Consistent with the theory of conditional bias, both ordinary kriging plans overestimate the true value in the high-grade case; however, in the low-grade case, the kriging estimates are also higher – contrary to what the theory of conditional bias would suggest – because the mean of the input distribution (0.97) is higher than almost all of the simulated grades; nevertheless, it is the integrated effect of all such local conditional bias investigations that realize a global conditional bias.

Discussion

The results are not surprising. Proper application of simulation, as we have described it, will always work when the simulated realizations make use of the same parameters (histogram, variogram, etc) as the true grades. Nevertheless, it is valuable to illustrate the correct way to use simulation in mine planning. The example would be more convincing with real data; however, it is uncommon to have sufficient true grades for proper testing.

This example has confirmed conventional wisdom, that is, ordinary kriging with a limited search routine results in good variance reproduction at the expense of conditional bias and ordinary kriging with a large search reduces conditional bias at the expense of smoothing. These are important because practitioners do not have the background, resources, or management support to switch to simulation using the methodology we have documented. The practitioner will, therefore, still be faced with the hard choice of improved global estimation of recoverable reserves or estimates with no conditional bias.

The procedure to base mine planning and recoverable reserves on validated $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ SMU estimates derived from conditional simulation is easily done. The UNIX scripts and FORTRAN codes that were used for this example are quite simple indeed. Commercial software vendors could implement some type of easy-to-use software with modest effort.

Alternative kriging implementations such as UC, DC, DK, MG, BG or MIK could also produce conditionally unbiased $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ block estimates, but these kriging-based methods are difficult to implement and make strong point-to-block distribution assumptions where conditional simulation assumes only that the conditioning data are Guassian. Moreover, conditional simulation is easy to implement, allows us to directly observe block values via averaging point scale values with the correct variability and can be used to effectively understand grade heterogeneities.

Synthetic examples such as this one assume "free selection" and "perfect selection", that is, the mining blocks can be selected independently and without error. This is not the case in practice. Instead, practical dig limits involve wasting some ore and processing some waste. A dilution factor is often considered or yet more simulation studies are conducted to evaluate the information effect and imperfect selection.

An Alternative Approach for Calculating the Probability of Ore and Ore Grade Estimates

We used the "sgsim" program from GSLIB to generate multiple geostsatistical realizations for calculating the probability of ore $\mathbf{P}_{ORE}$ and ore grade $\mathbf{Z}_{ORE}$ SMU estimates; however, LU decomposition simulation could also be used. In this approach, a mining block is discretized into a number of nodes. A covariance matrix consisting of data-to-data, node-to-node and node-to-data covariances is constructed and decomposed into lower and upper triangular matrices. Multiplying the lower triangular matrix by a column vector composed of conditioning data and random numbers will simultaneously simulate the nodes. Averaging the simulated node values and repeating the procedure will provide multiple realizations of the block grade. These block values can be used to calculate $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ estimates. The rest of the blocks are then visited similarly. The GSLIB package contains a program called "lusim" that implements the LU decomposition simulation method.

Any small subset of the study area can be discretized; there is no need to define a regular SMU size and geometry. The problems with change of support are thus handled in a flexible way. The idea of abandoning the SMU concept altogether was proposed by Rossi (2000).

LU decomposition simulation is particularly appropriate when the number of conditioning data plus the number of nodes to be simulated is small (<1000) and the number of realizations required is large. Implementation is straightforward for the practitioner since the parameters are similar to kriging. CPU demand is also similar to kriging – it takes approximately 20% more time than ordinary kriging, Davis (2001). On the down side, the LU decomposition method does not generate joint uncertainty at all locations (between blocks) simultaneously; only the "within-block" variability is modeled correctly.

The LU decomposition approach is an efficient alternative to calculate the $\mathbf{P}_{ORE}$ and $\mathbf{Z}_{ORE}$ block estimates. This LU decomposition approach to mine planning and recoverable reserve estimation has been successfully documented and implemented by Ed Isaaks and Bruce Davis, Davis (2001).

Conclusion

Basing mine plans on probability of ore and ore grade estimates calculated from multiple geostatistical realizations effectively solves the longstanding problem of conditional bias. Geostatistical simulation, the theory of block averaging, the calculation of probability of ore and ore grade are consistent with recoverable reserve estimation for mine planning. Implementation is straightforward and the procedure can be run on low-level PC's available at virtually every mine site.

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Appendix: Details of Conditional Bias For Ordinary Kriging

To check the conditional bias of a set of estimates one must plot the true values versus the estimates, both at the same support. Conditional bias is then quantified by how much the slope of the regression line deviates from unity (slopes less than one indicate over-estimation in high-grade areas and under-estimation in low-grade areas). This appendix discusses the factors that affect conditional bias. Our estimates are created from ordinary kriging.

The slope of the regression line depends on many parameters such as the histogram, variogram, mining support, sample spacing, cutoff grade, etc. These are often set in practice, but a numerical laboratory such as the one created for the example in this paper could be setup to explore the relationship between these variables and conditional bias. We do not even attempt to show all relationships, but there are a few important ones.

Figure 14 shows the regression slope versus the kriging variance for four possible sample spacings. As the sample spacing increases, the kriging variance increases (it becomes harder to estimate) and the regression slope decreases, that is, the amount of conditional bias increases. In Figure 15, the regression slope is plotted against the number of maximum conditioning data used for kriging. Consistent to what was shown in the example before, more data retained results in less conditional bias.

Ordinary kriging requires an assumption of local stationarity within the span of subsequent local search neighborhoods. These local assumptions of stationarity are much less severe than for simple kriging where the entire deposit is assumed stationary; however, retaining a large number of data in the ordinary kriging plan to reduce uncertainty and conditional bias will make the stationarity assumption more severe.

D.G. Krige has gone to great lengths exploring the quantitative and qualitative effect different factors have on conditional bias. The conditional bias of kriging (especially ordinary kriging) has been well researched and is well understood.

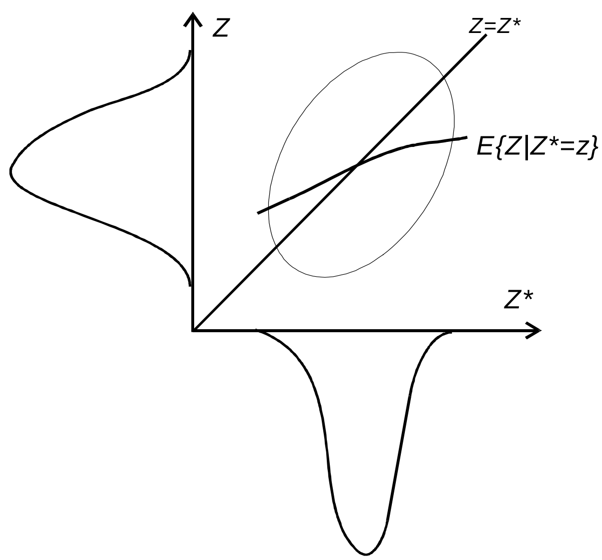

Figure 1 – A Schematic Illustration of Conditional Bias. The estimates (z^*) are on the abscissa axis and the true grades (Z_V) are on the ordinate axis. The distribution of true grades (left) has more variability than the distribution of estimated grades (bottom) due to the smoothing effect of kriging. The ellipse represents the scatter of paired true and estimated grades and the curved line shows conditional bias in the form of a conditional expectation curve, $E\{Z_V \mid Z^*_V = z\}$.

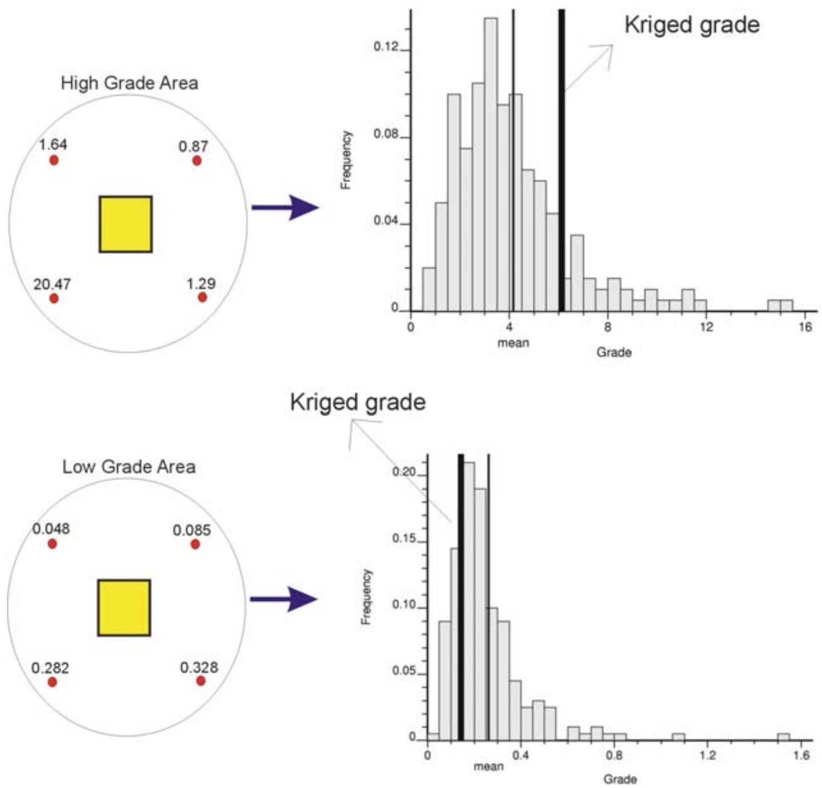

Figure 2 - Two Cases of Conditional Bias. The sketches to the left show the data configuration and a central block being estimated (the block is 10 m on a side). The histograms to the right are the distributions of true grades conditional to the 20m spaced sample data. The heavy vertical line is the kriged grade and the light vertical line is the mean grade. The kriged grade is too high in the high-grade case and too low in the low-grade case.

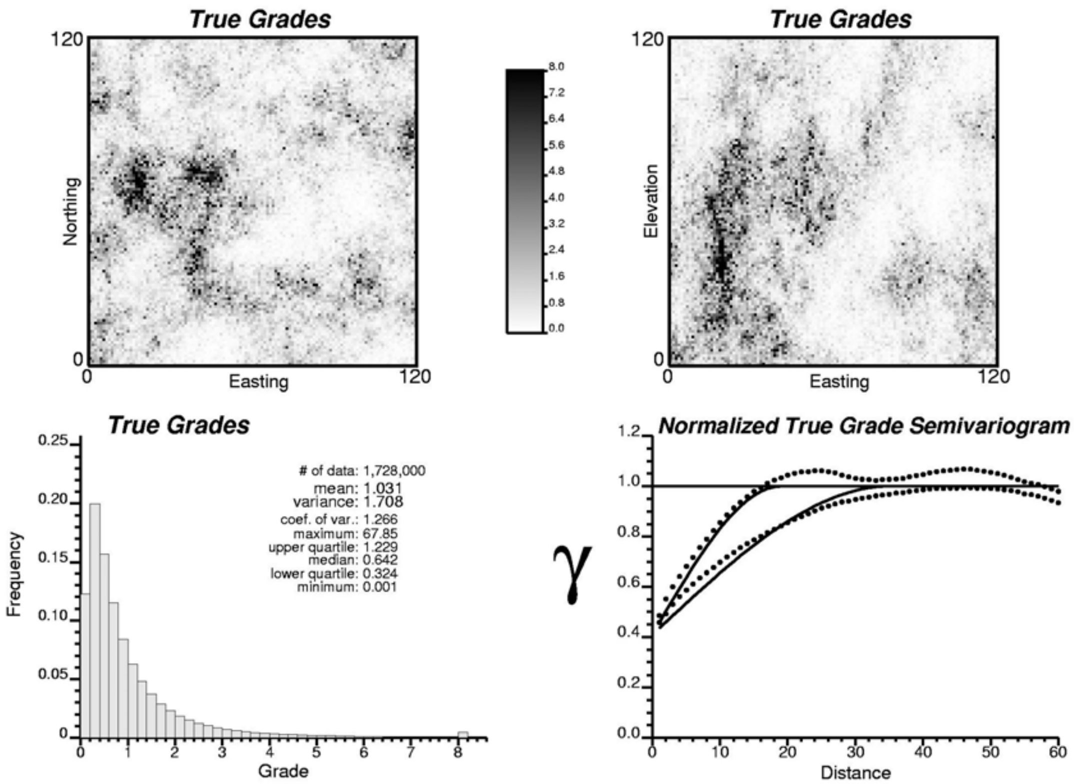

Figure 3 – Reference Data Model. Central XY and XZ slices through the 1m spaced true grade model are shown. The true grades are lognormally distributed with a mean of 1.0 and variance of 1.7 as shown by the histogram. The normal score horizontal and vertical semivariogram is calculated (shown as points) from the true values plotted on top of the model (shown as solid lines).

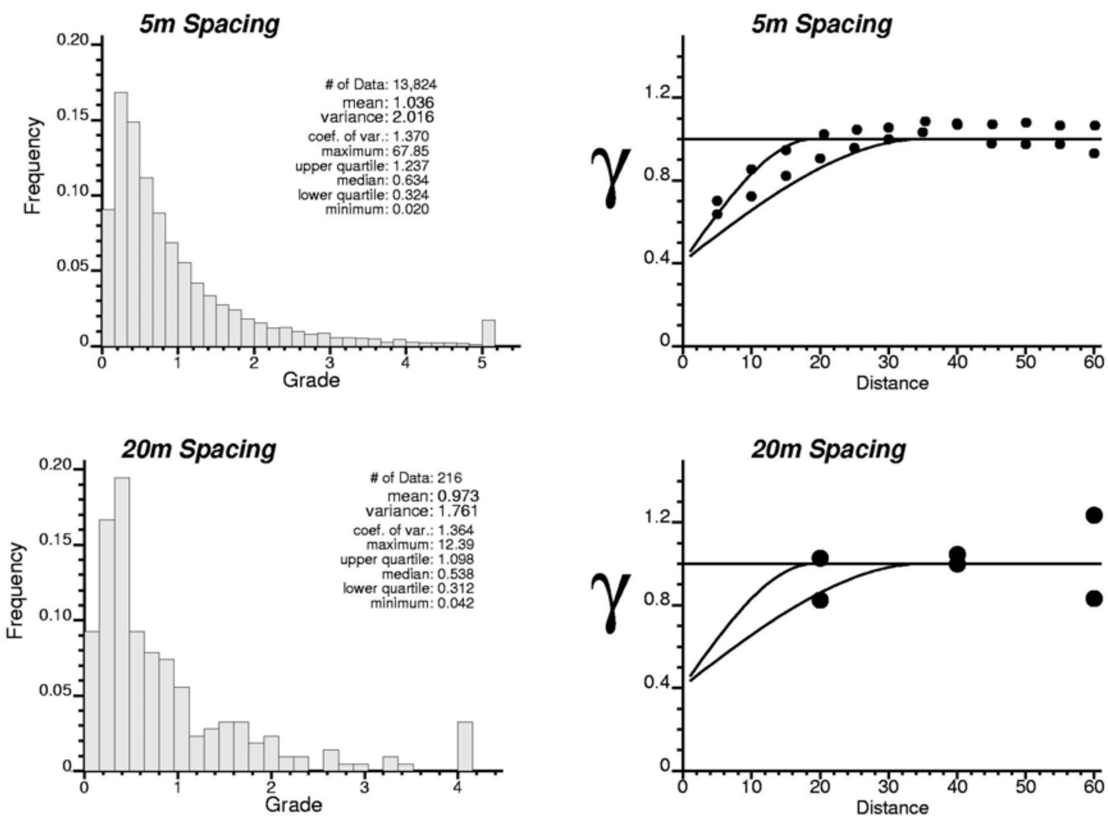

Figure 4 – Sample Data. The histogram and calculated variogram (points) vs. model variogram (solid lines) for the 5m and 20m spaced sample data sets.

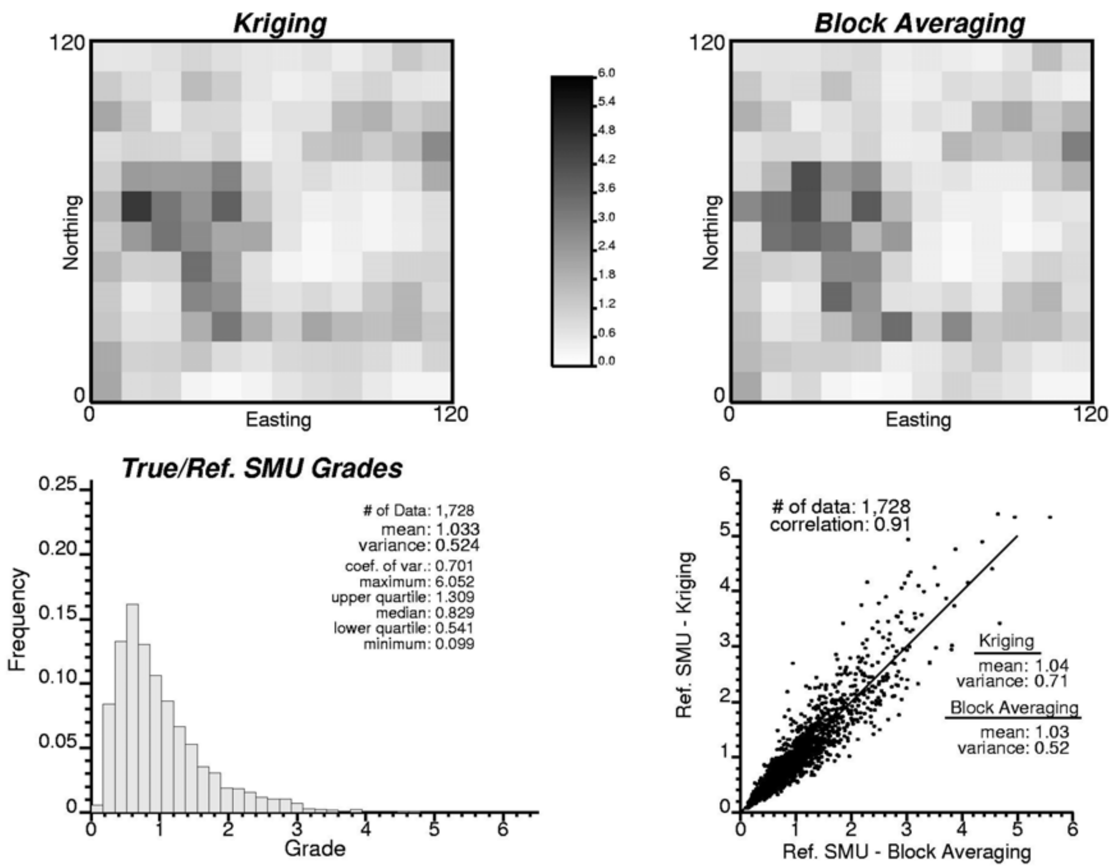

Figure 5 – SMU Reference Grades. A central XY slice (subsequent maps will be presented in the same XY orientation and at the same central level) through the true grade SMU model calculated by (1) ordinary kriging with the 5m grid of sample data and by (2) block averaging the 1m spaced exhaustive true grades. The strong correlation between the two methods is shown in the cross-plot of paired true SMU grades created from both methods. The histogram of true SMU grades created from block averaging is also shown.

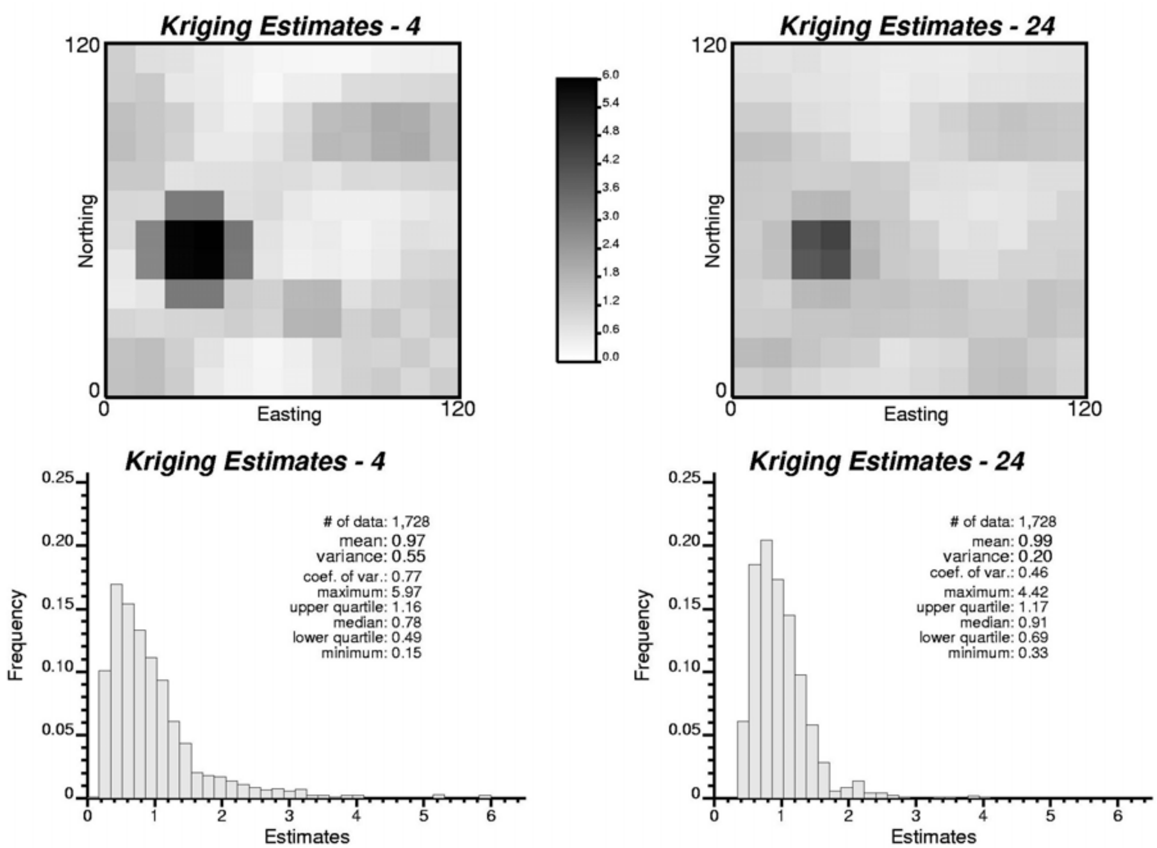

Figure 6 – Kriging Results. The kriging results with 4 and 24 maximum conditioning data retained. When compared to the true SMU grades created from block averaging (see Figure 5) the correct dispersion variance (0.52) is underestimated when kriging with a maximum of 24 conditioning data (0.20) and is well reproduced when kriging with 4 maximum conditioning data (0.55).

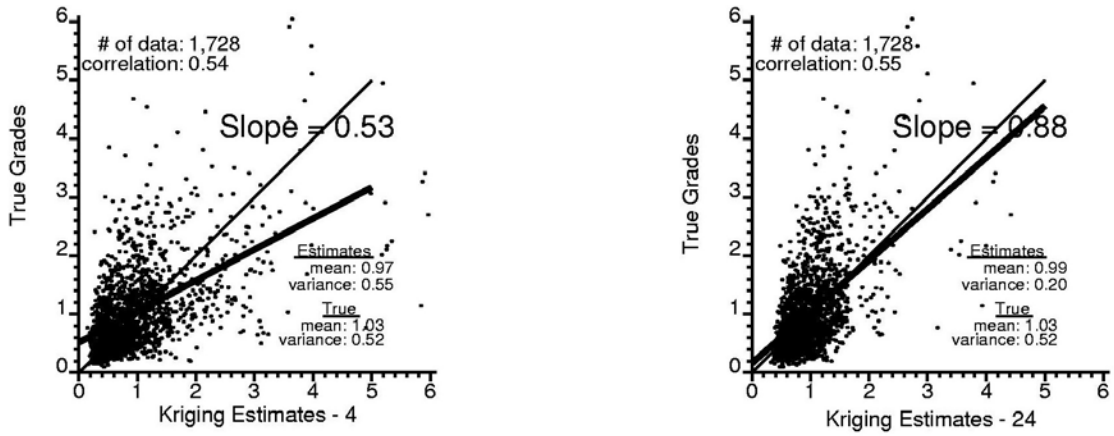

Figure 7 – Conditional Bias of Ordinary Kriging. The cross-plots of the true SMU grades created from block averaging vs. the ordinary kriging estimates with 4 and 24 maximum conditioning data retained (see Figure 6). Observe the severe conditional bias (regression slope = 0.53) and good variance reproduction when kriging with 4 maximum conditioning data and the mild conditional bias (regression slope = 0.88) and severe underestimation of the underlying variance when kriging with 24 maximum data.

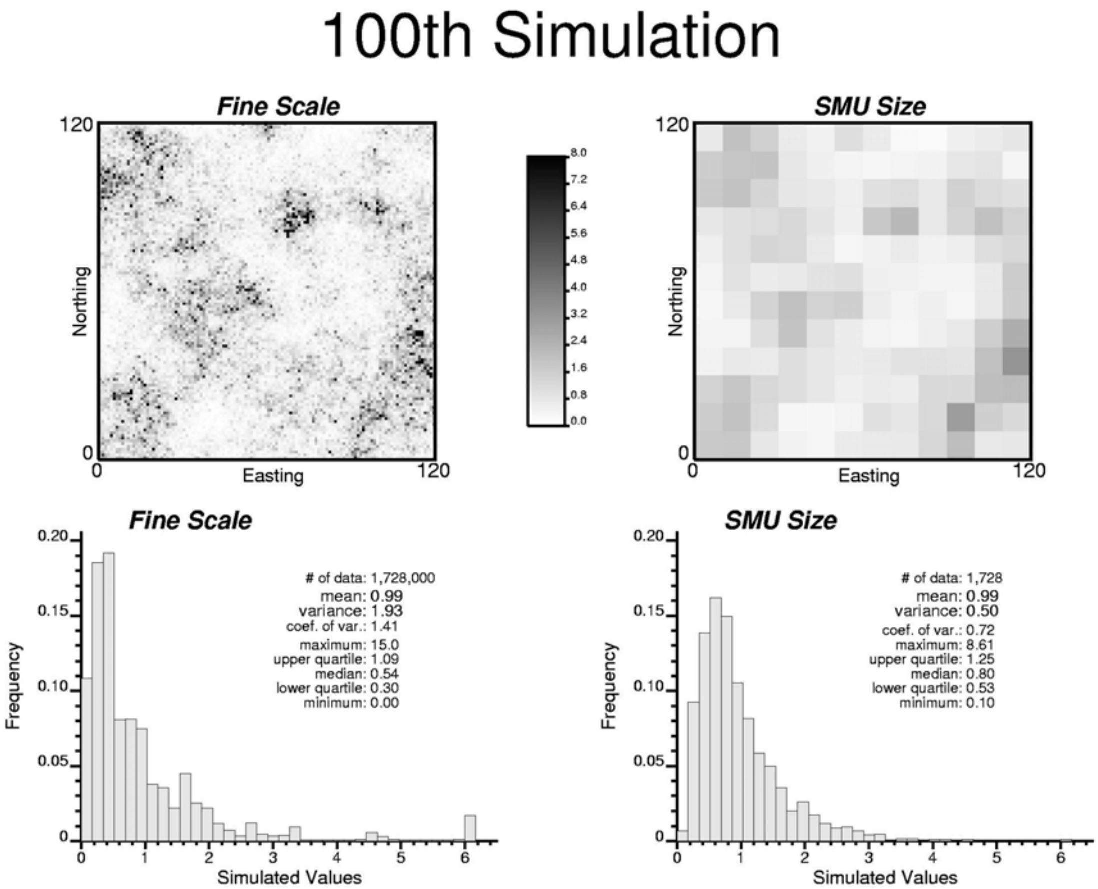

Figure 8 – Simulation Results. The fine-scale simulation and block averaging procedure for the last of 100 realizations. Notice the similarities with the true exhaustive data set (see Figure 3) and the true block averaged SMU grades (see Figure 5).

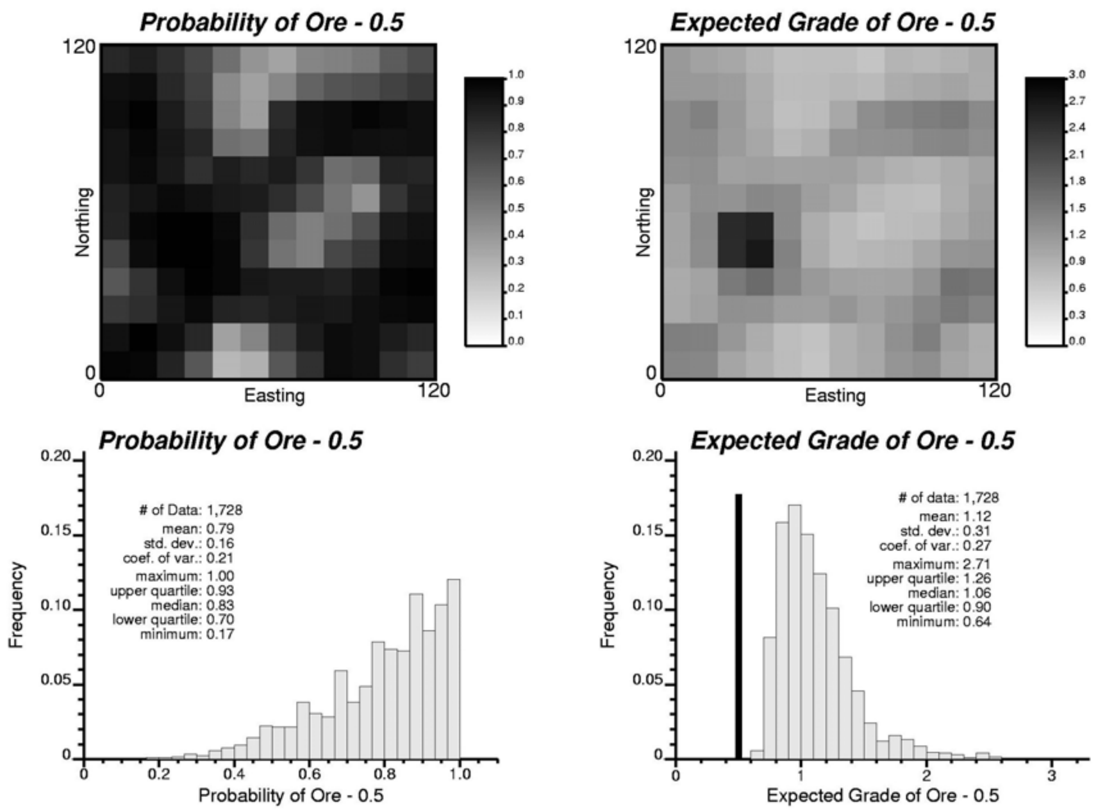

Figure 9 – Mine Planning. Probability of ore and ore grade estimates using a 0.5 base-case cutoff grade. Notice the high probabilities to be ore and the high expected ore grades in the high-grade areas. The minimum of the probabilities is 0.17 and the dark vertical line (right) is the cutoff grade.

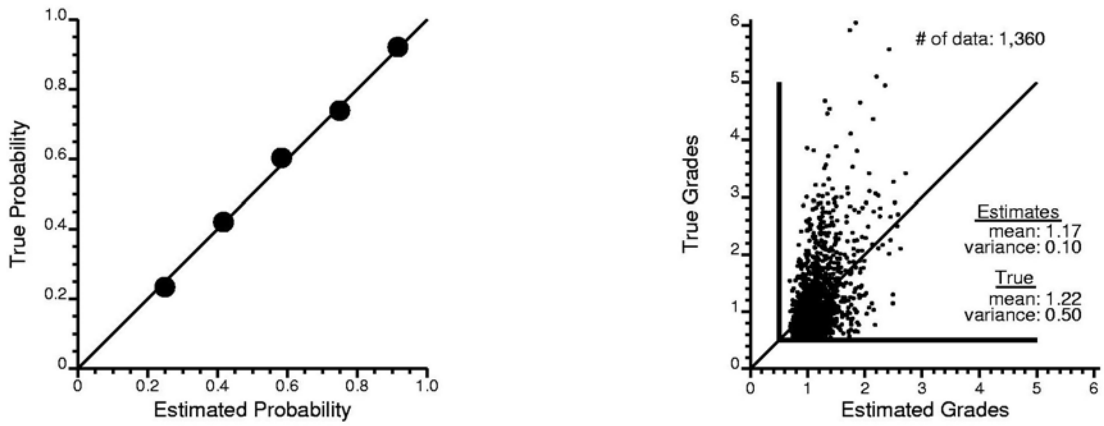

Figure 10 – Conditional Bias of Simulation. The cross plot of true versus estimated probability of ore and true vs. estimated ore grades for the 0.5 base-case cutoff. Notice the perfect probability correlations and good agreement of ore grade means.

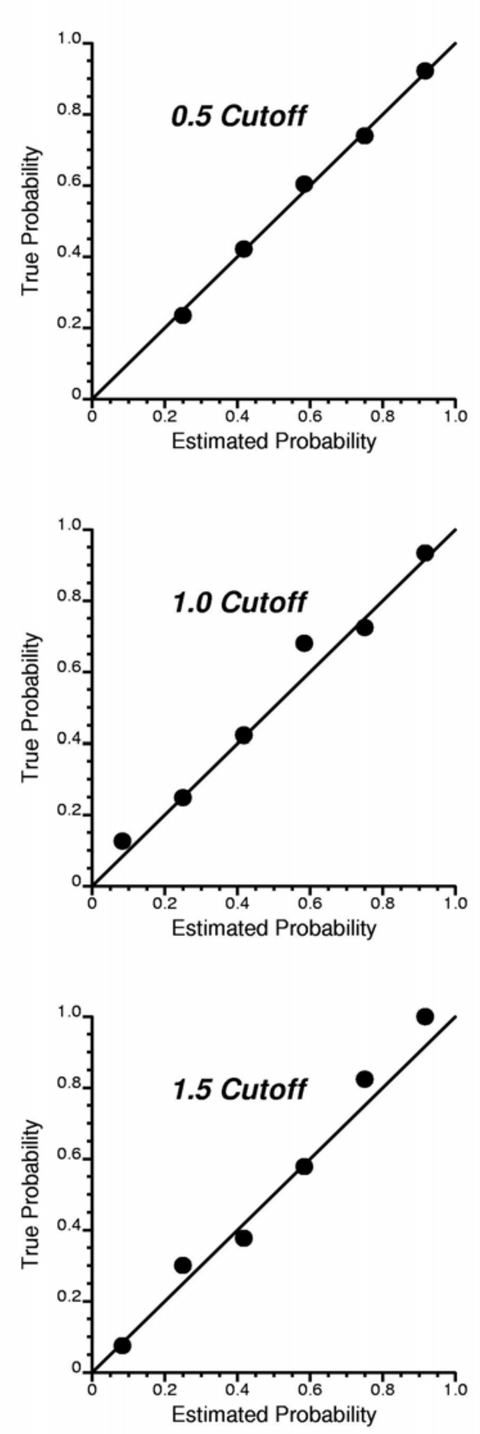

Figure 11 – Sensitivity to Cutoff. The conditional bias of probability of ore estimates for cutoff grades at 0.5, 1.0 and 1.5 (below, at and above the mean). Perfect correlation persists.

		$Z_t \mid Z_t > cutoff$	$Z^* \mid Z^* > \text{cutoff}$
Cutoff grades	**0.5**	**1.22**	**1.17**
	1.0	**1.69**	**1.65**
	1.5	**2.24**	**2.20**

Figure 12 – Sensitivity to Cutoff. The conditional bias of expected grade of ore for cutoffs at 0.5, 1.0 and 1.5. The cross plots of true vs. estimated grades of ore are summarized in the table above. The agreement of means persists.

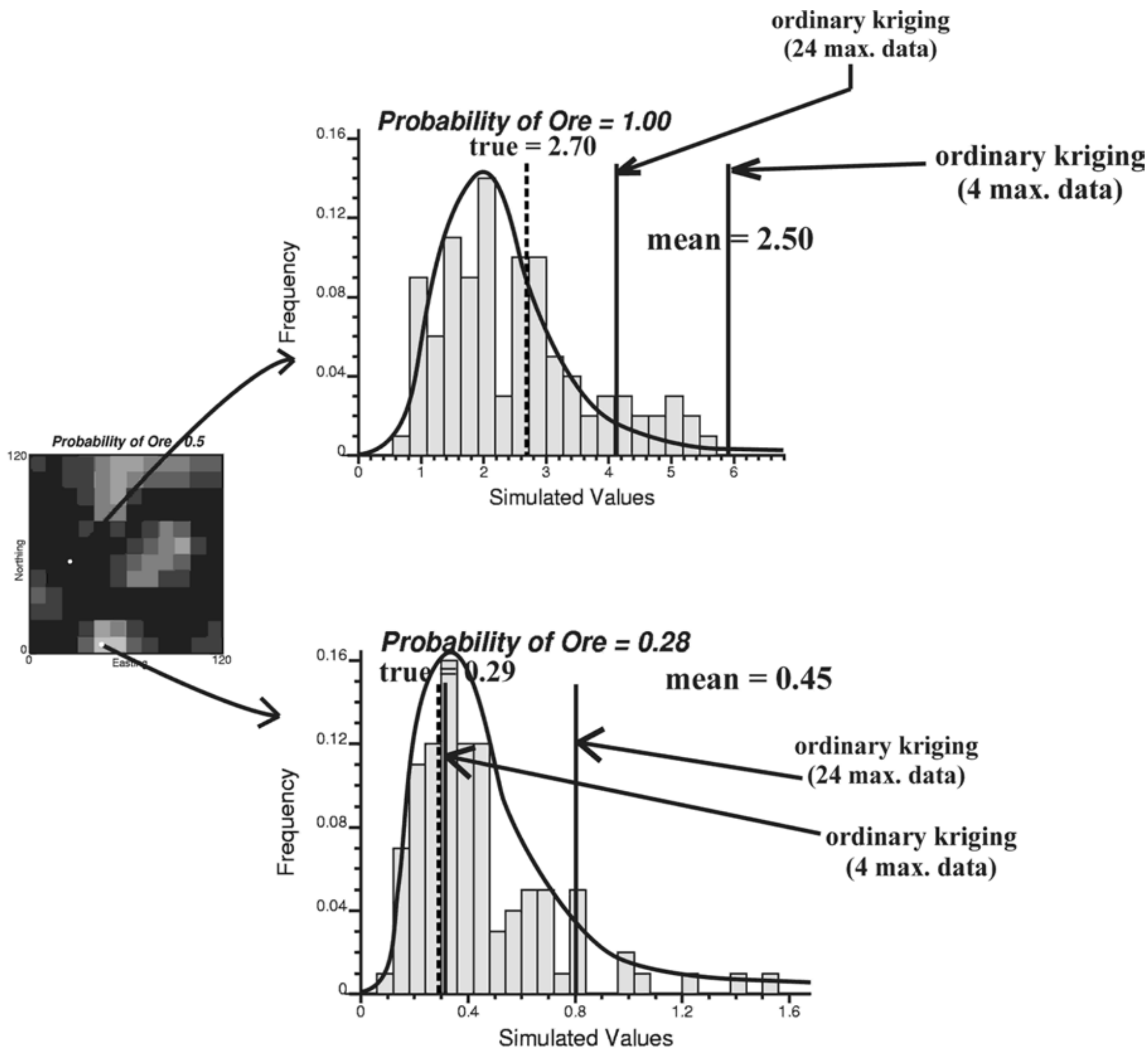

Figure 13 – Local Validation. Local conditional bias is investigated at the locations where the highest and lowest probabilities of ore estimates exist on the central XY slice (left). The distribution of simulated grades is shown at each location along with two solid vertical lines representing ordinary kriging estimates using 4 and 24 maximum conditioning data and a broken vertical line representing the true block averaged value. The kriging estimates are higher than the true value at both locations; therefore, the theory of conditional bias is consistent for only the high-grade location.

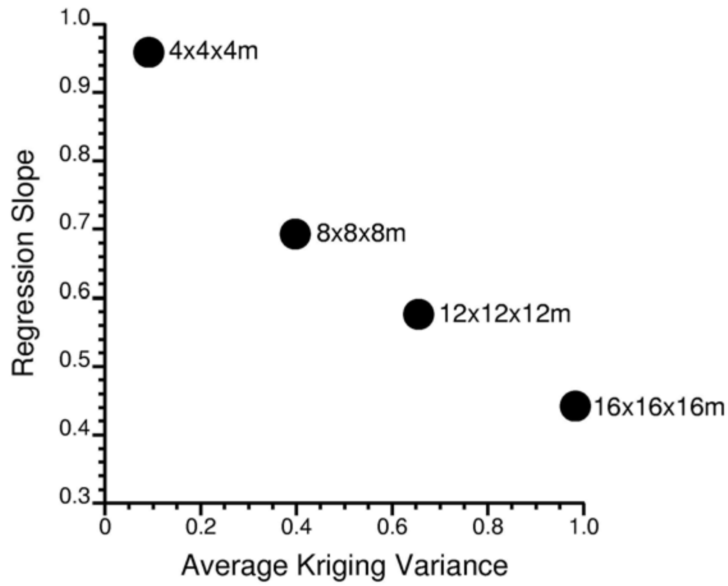

Figure 14 – Sample Spacing. The relationship between conditional bias and sample spacing. As the sample spacing increases the kriging variance increases (it becomes harder to estimate) and the slope of the regression line decreases, which imparts more conditional bias.

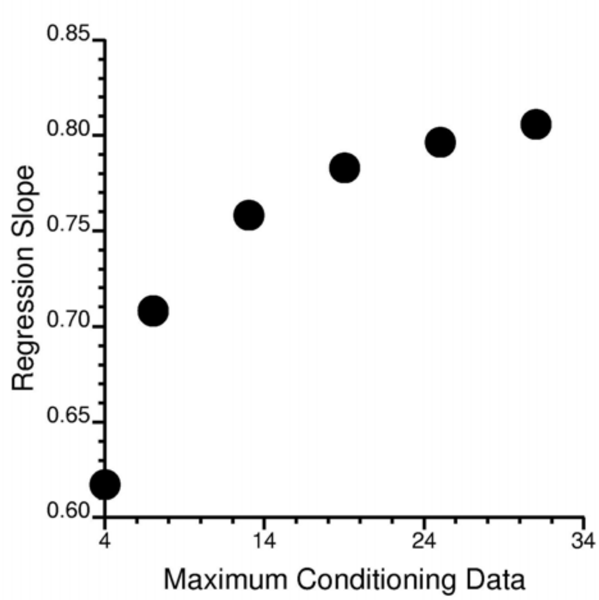

Figure 15 – Maximum Conditioning Data. The relationship between conditional bias and number of maximum conditioning data retained for ordinary kriging. As the maximum number of conditioning data increases, the slope of the regression line increases indicating less conditional bias.