A Review of Separable Spatiotemporal Models of Regionalization

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In order to characterize and predict space-time phenomena, it is necessary to recognize the spatiotemporal covariance function which should be used in constructing the kriging system of equations. This note provides a summary of separable spatiotemporal models of regionalization. Specifically, six models are reviewed. A small synthetic example is constructed based on satellite imagery collected over regular time periods to illustrate the calculation and visualization of these spatiotemporal variograms.

Introduction

Common geostatistical applications focus on the spatial correlations of a natural phenomena; however, there is nothing inherently exclusive about a spatially variable attribute that lends itself to geostatistics. The use of geostatistics to model a phenomenon that varies with space *and* time, that is a spatiotemporal phenomenon, is not new to seasoned geostatisticians. Nevertheless, the requirements and considerations for adapting common geostatistical measures to address these problems are not well-known to most geomodelers.

Consider that data are collected in some spatial configuration over multiple time periods (see Figure 1). We can imagine that for each time period, there is a spatial configuration that is common to consider in conventional geostatistical applications. There is no requirement that data at the same spatial locations are recorded in regular time intervals. In fact, they may be recorded at different locations in irregular time steps. The schematic shown in Figure 1 displays data collected at the same n_s locations, as one might expect if these locations represent monitoring stations, in n_t regular time intervals.

Suppose that $Z(\mathbf{u},t)$ denotes a spatiotemporal process observed at different locations and time steps. This data set is treated as a non-random sample from a realization of a space-time random field

$$Z(\mathbf{u},t);\ \mathbf{u} \in A, t \in T$$

Where u represents a location vector in domain *A*, and *t* is a time period taken over *T* time periods, therefore $Z(\mathbf{u},t)$ is a spatio-temporal random variable (STRV). The ultimate objective is the optimal estimation and prediction of *Z* at unsampled locations in space and time subject to *n* observations.

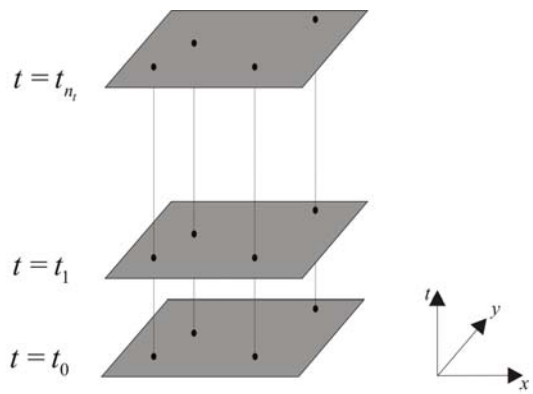

Figure 1: Example for Spatiotemporal Data in R^2 x T Space, in this schematic, the location of data is fixed with respect to time.

Suppose that the first and second moment of Z exists and Z can be written as a decomposition.

$$Z(\mathbf{u},t) = m(\mathbf{u},t) + Y(\mathbf{u},t)$$

where $m(\mathbf{u},t)$ is a mean or trend function of STRV Z, and $Y(\mathbf{u},t)$ is a residual STRV with $E\{Y(\mathbf{u},t)\}=0$. The spatiotemporal covariance function corresponding to this STRV Z is equivalent to defining the covariance function of $Y(\mathbf{u},t)$:

$$C_{ST}(\mathbf{h}_s,\mathbf{h}_t) = Cov\{Y(\mathbf{u}+\mathbf{h}_s, t+\mathbf{h}_t), Y(\mathbf{u},t)\}$$

where $\mathbf{h}_s$ is spatial lag vector, $\mathbf{h}_s \in A$; and $\mathbf{h}_t$ is temporal lag vector, $\mathbf{h}_t \in T$. The variogram model associated to this STRV is given as

$$2\gamma_{ST}(\mathbf{h}_s,\mathbf{h}_t) = E\left\{\left[Y(\mathbf{u},t) - Y(\mathbf{u}+\mathbf{h}_s, t+\mathbf{h}_t)\right]^2\right\}$$

Choosing a valid spatiotemporal covariance and variogram models is one of the issues related to spatiotemporal geostatistics for space-time data. There are two general approaches to extend the spatial variogram or covariance into space-time: (1) treat space-time as simply a higher dimension, or (2) separate space and time. The following section reviews some of the spatiotemporal models of regionalization that have been developed specifically for spatiotemporal phenomenon by using the second approach.

Models for Spatiotemporal Variograms

Six models are examined in this section; they are reviewed in chronological order. All the models presented below are based on the covariance; the equivalent variogram forms of these models are also shown but require an assumption of first and second order stationarity. These stationarity assumptions are also used in calculating the experimental marginal spatiotemporal variograms which are later discussed and calculated for a sample space-time data. The relationship between the spatiotemporal covariance, variance and variogram based on these stationarity assumptions are

$$\gamma_{ST}(\mathbf{h}_s,\mathbf{h}_t) = C_{ST}(0,0) - C_{ST}(\mathbf{h}_s,\mathbf{h}_t)$$

1. *Sum Model (Rouhani and Hall, 1989):* In this type of model we assume that the spatiotemporal covariance is the summation of spatial and temporal covariances, therefore:

$$C_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=C_S\left(\mathbf{h}_s\right)+C_T\left(\mathbf{h}_t\right)$$

where $C_S(\mathbf{h_s})$ is the spatial covariance and in practice can be assumed proportional to the spatiotemporal covariance at $\mathbf{h}_t=0$, that is $C_{ST}(\mathbf{h_s},0)$, and $C_T(\mathbf{h_t})$ is the temporal covariance and can be assumed proportional to the spatiotemporal covariance at $\mathbf{h}_s=0$, that is, $C_{ST}(0,\mathbf{h_t})$. These two covariances are also referred to as the marginal covariances. Alternatively, we can consider the variogram equivalent of this model:

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma_S\left(\mathbf{h}_s\right)+\gamma_T\left(\mathbf{h}_t\right)$$

where $\gamma_S\left(\mathbf{h}_s\right)$ is the spatial variogram and is calculated as the spatiotemporal variogram at $\mathbf{h}_t=0$ which is $\gamma_{ST}\left(\mathbf{h}_s,0\right)$), and $\gamma_T\left(\mathbf{h}_t\right)$ is the temporal variogram and is calculated as the spatiotemporal variogram at $\mathbf{h}_s=0$ which is $\gamma_{ST}\left(0,\mathbf{h}_t\right)$. Similar to the covariance, these two variograms are also called the marginal variograms. Two other marginal spatiotemporal variograms can also be defined but they are not used in evaluating the spatiotemporal models of regionalizations, they are $\gamma_{ST}\left(\mathbf{h}_s,\infty\right)$ which is the spatiotemporal variogram when $\mathbf{h}_t$ approaches infinity and $\gamma_{ST}\left(\infty,\mathbf{h}_t\right)$ which is the spatiotemporal variogram when $\mathbf{h}_s$ approaches infinity.

   Although the separate spatial and temporal covariance functions are positive semi-definite, Myers and Journel (1990) showed by example that use of this model can result in a singular system of kriging equations, under certain data configurations.

2. *Metric Model (Dimitrakopoulos and Luo, 1994):* If $C(\mathbf{h})$ is strictly positive definite on $R^d\times T$ and $\gamma(\mathbf{h})$ is strictly conditionally negative definite on $R^d\times T$ then the metric model for covariance and variogram is as below:

$$C_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=C\left(\left|\mathbf{h}_s\right|+a\left|\mathbf{h}_t\right|\right)$$

   and

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma\left(\left|\mathbf{h}_s\right|+a\left|\mathbf{h}_t\right|\right)$$

   where a is the ratio of the geometric anisotropic range.

3. *Alternative Metric Model (Dimitrakopoulos and Luo, 1994):* Dimitrakopoulos and Luo (1994) suggested that, $\left(\left|\mathbf{h}_s\right|^2+a\left|\mathbf{h}_t\right|^2\right)$ can be used instead of using $\left(\left|\mathbf{h}_s\right|+a\left|\mathbf{h}_t\right|\right)$ as a distance function on $R^d\times T$ in the metric model:

$$C_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=C\left(\left|\mathbf{h}_s\right|^2+a\left|\mathbf{h}_t\right|^2\right)$$

   and

$$\gamma_{ST}(\mathbf{h}_s, \mathbf{h}_t) = \gamma\left(|\mathbf{h}_s|^2 + a|\mathbf{h}_t|^2\right)$$

In terms of the topology theses two metric models are equivalent (Myers, 2004).

4. *Sum-Metric Model:* This model is the combination of the sum and metric models and takes the following form

$$C_{ST}(\mathbf{h}_s, \mathbf{h}_t) = C_S(\mathbf{h}_s) + C_T(\mathbf{h}_t) + C\left(|\mathbf{h}_s| + a|\mathbf{h}_t|\right)$$

or

$$C_{ST}(\mathbf{h}_s, \mathbf{h}_t) = C_S(\mathbf{h}_s) + C_T(\mathbf{h}_t) + C\left(|\mathbf{h}_s|^2 + a|\mathbf{h}_t|^2\right)$$

And in the variogram forms they are reduced to

$$\gamma_{ST}(\mathbf{h}_s, \mathbf{h}_t) = \gamma_S(\mathbf{h}_s) + \gamma_T(\mathbf{h}_t) + \gamma\left(|\mathbf{h}_s| + a|\mathbf{h}_t|\right)$$

or

$$\gamma_{ST}(\mathbf{h}_s, \mathbf{h}_t) = \gamma_S(\mathbf{h}_s) + \gamma_T(\mathbf{h}_t) + \gamma\left(|\mathbf{h}_s|^2 + a|\mathbf{h}_t|^2\right)$$

The sum-metric model was proposed to remove the semi-definiteness condition of the sum model. It can be proved that the sum-metric model is strictly conditionally definite.

5. *Product Model (Cesare et al. 2001):* The product model assumes that the spatiotemporal covariance is the multiplication of both marginal covariances (spatial and temporal)

$$C_{ST}(\mathbf{h}_s, \mathbf{h}_t) = kC_S(\mathbf{h}_s)C_T(\mathbf{h}_t)$$

where $C_{ST}(0,0)$ is the global sill calculated as the variance of all the space-time data taken together; $C_S(0)$ is the average covariance of the spatial data taken over each time period, and is referred to as the spatial sill; $C_T(0)$ is the average covariance of the data values at each location taken over all time periods, and is referred to as the temporal sill; and k is a constant calculated as:

$$k = \frac{C_{ST}(0,0)}{C_S(0)C_T(0)}$$

Cesare et.al. (2001) showed that the corresponding variogram model under this construction is

$$\gamma_{ST}(\mathbf{h}_s, \mathbf{h}_t) = kC_T(0)\gamma_S(\mathbf{h}_s) + kC_S(0)\gamma_T(\mathbf{h}_t) - k\gamma_S(\mathbf{h}_s)\gamma_{ST}(\mathbf{h}_t)$$

This spatiotemporal covariance is positive definite if both spatial and temporal covariances are positive definite. In the product model for any two fixed spatial lags $\mathbf{h}_1$ and $\mathbf{h}_2$, the corresponding spatiotemporal covariances are proportional to each other, that is

$$C_{ST}(h_1, \mathbf{h}_t) \quad \propto \quad C_{ST}(h_2, \mathbf{h}_t)$$

6. *Product-Sum Model (Cesare et al. 2001):* This model is a linear combination of the product and sum models:

$$C_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=k_1C_S\left(\mathbf{h}_s\right)C_T\left(\mathbf{h}_t\right)+k_2C_S\left(\mathbf{h}_s\right)+k_3C_T\left(\mathbf{h}_t\right)$$

And in the variogram form it is

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\left[k_1C_T\left(0\right)+k_2\right]\gamma_S\left(\mathbf{h}_s\right)+\left[k_1C_S\left(0\right)+k_3\right]\gamma_T\left(\mathbf{h}_t\right)-k_1\gamma_S\left(\mathbf{h}_s\right)\gamma_T\left(\mathbf{h}_t\right)$$

To calculate the coefficients two constraints are imposed: (1) set the first two coefficients equal to 1.0, and (2) the third equation uses the amount of the covariance model at $\mathbf{h}_s$=0 and $\mathbf{h}_t$=0. Thus the constants k_1, k_2 and k_3 are calculated as

$$k_1=\frac{C_S\left(0\right)+C_T\left(0\right)-C_{ST}\left(0,0\right)}{C_S\left(0\right)C_T\left(0\right)}$$

$$k_2=\frac{C_{ST}\left(0,0\right)-C_T\left(0\right)}{C_S\left(0\right)}$$

$$k_3=\frac{C_{ST}\left(0,0\right)-C_S\left(0\right)}{C_T\left(0\right)}$$

These constraints impose a form of symmetry between the impact of the spatial component and the temporal correlation component therefore this is one of the disadvantages of using this kind of model.

If we assumed that the product-sum model for the spatiotemporal variogram is as below (Cesare et.al., 2001):

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma_{ST}\left(\mathbf{h}_s,0\right)+\gamma_{ST}\left(0,\mathbf{h}_t\right)-k_1\,\gamma_{ST}\left(\mathbf{h}_s,0\right)\gamma_{ST}\left(0,\mathbf{h}_t\right)$$

Iaco et al. (2001) proved that the necessary and sufficient condition for k_1 is

$$k_1=\frac{C_S\left(0\right)+C_T\left(0\right)-C_{ST}\left(0,0\right)}{C_S\left(0\right)C_T\left(0\right)}$$

and

$$0<k_1\leq\frac{1}{\max\left\{C_S\left(0\right),\,C_T\left(0\right)\right\}}$$

This is a valid model in space-time, if the separate space and time models are valid and $k_1>0$; further, k_2 and k_3 must be non-negative while k_1 must be strictly positive. (Cesare et al. 2001).

One of the advantages of both the product and product-sum models is that these models are easily fit by the using the marginal variograms $\gamma_{ST}\left(\mathbf{h}_s,0\right)$ and $\gamma_{ST}\left(0,\mathbf{h}_t\right)$. The marginal variograms can be estimated by two ways under the second order stationarity assumption; first calculating the spatial variograms for each time step and then averaging over time and second calculating the temporal variograms for each data location and averaging over space.

Almost all of the spatiotemporal covariance models presented above are separable, that is, they do not account for the interaction between time and space. Cressie and Huang (1999) presented some classes of nonseparable spatiotemporal variograms. These nonseparable models contain the spatial and temporal components implicitly coupled with each other; however, they are beyond the scope of this study.

The next section considers application of some of these models to a synthetic data set based on collected satellite images sampled over 36 regular intervals. Specifically, the sum, product and sum-product models are examined in greater detail.

Synthetic Spatiotemporal Case:

Satellite images showing the movement of the clouds off the coast of Florida are used for this example of a spatiotemporal process (see Figure 2). For this synthetic case, an image is taken at 36 regular time intervals. Therefore there are 36 images and each of them has 400 by 400 grid nodes. To create a spatiotemporal data set, 200 'samples' are drawn randomly and the colors of these points are converted to RGB (red, green and blue) code. The RGB code represents the color of the pixel. It consists of three numbers that are between 0 and 255. For example the black color has the RGB code of (0,0,0) while the white color is (255,255,255). This gives a 2D data set for each time step. Further, note that the locations of the 200 samples are the same with respect to time. Figure 3 shows the satellite images for the base case ($t = t_0$) and for the 26th time step ($t = t_{25}$) and the red value (R code) as a spatiotemporal variable.

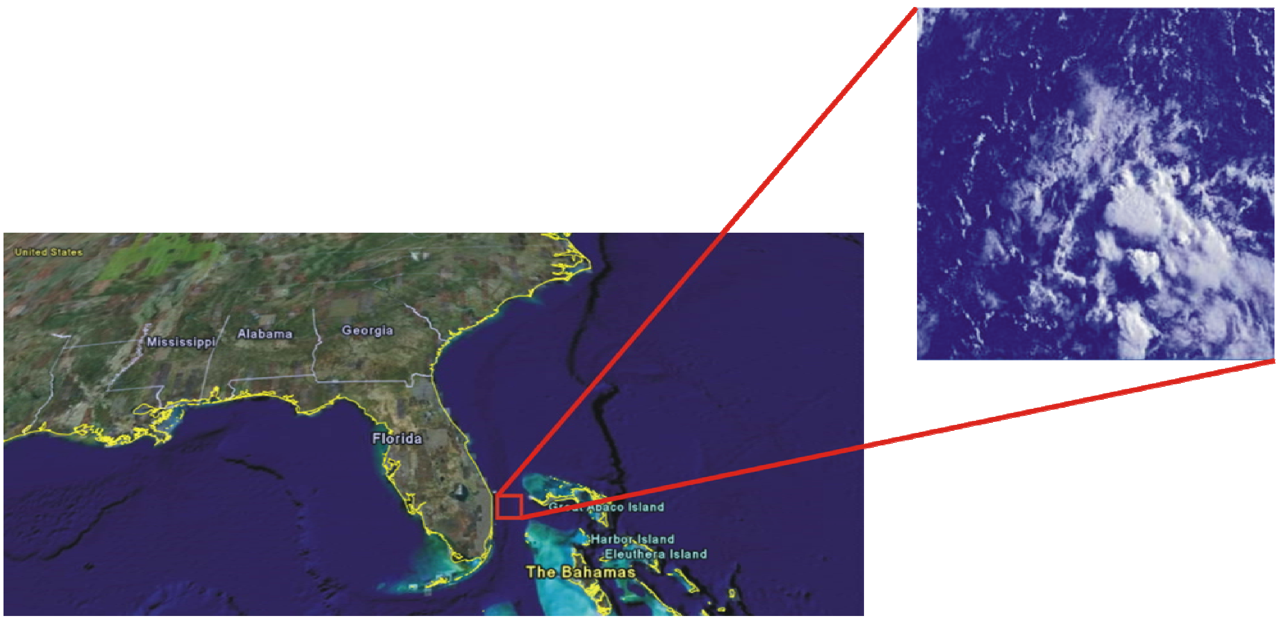

Figure 2: Location of satellite images collected at regular time intervals over a 12 hour period. (Source: http://www.wunderground.com and http://earth.google.com, 2006)

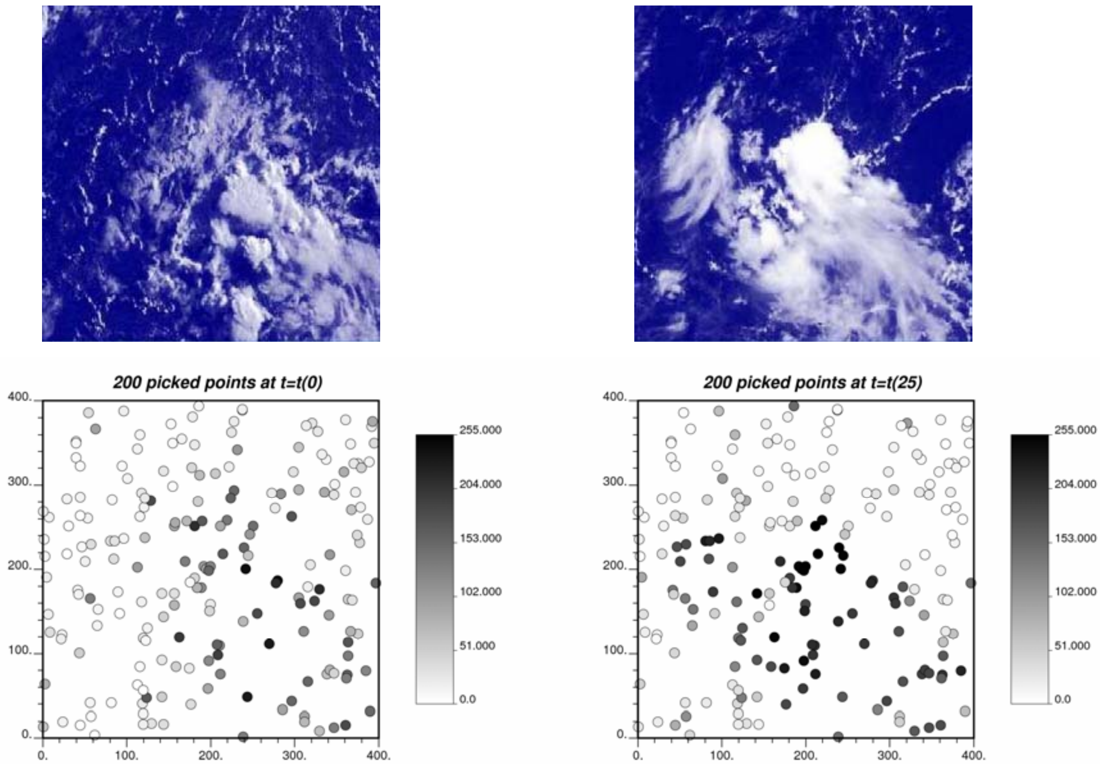

Figure 3: The satellite images (top row) at $t = t_0$ (left) and at $t = t_{25}$ (right), and the 200 samples drawn from these two time steps showing the R (or red) code associated to the images (bottom row).

For this kind of spatiotemporal process we can define two kinds of means as well as variances. The first kind of mean is the mean over space which varies with time, and second is the mean over time which varies with space. To calculate the mean over space as a function of time, $m_{\mathbf{u}}(t)$, an average of the spatial samples is taken at each time step, therefore for each time step there is a spatial mean value; to calculate the mean over time as a function of space, $m_t(\mathbf{u})$, an average is taken at each location over time therefore for each of the 200 samples there is a mean which is taken over all 36 time intervals. The definitions of the variances are the same as the means. They are the variance over space as a function of time, $\sigma^2{}_{\mathbf{u}}(t)$, and the variance over time as a function of space, $\sigma^2{}_t(\mathbf{u})$. Figure 4 shows the scatter plot of the $m_{\mathbf{u}}(t)$ and $\sigma^2{}_{\mathbf{u}}(t)$ versus time; Figure 5 shows the spatial locations of $m_t(\mathbf{u})$ and $\sigma^2{}_t(\mathbf{u})$. It can be easily seen from Figure 4 that the shape of the scatter plots for mean and variance versus time are approximately the same. It shows the proportional effect. To see better the proportional effect, the scatter plots of standard deviation versus mean are plotted in Figure 6.

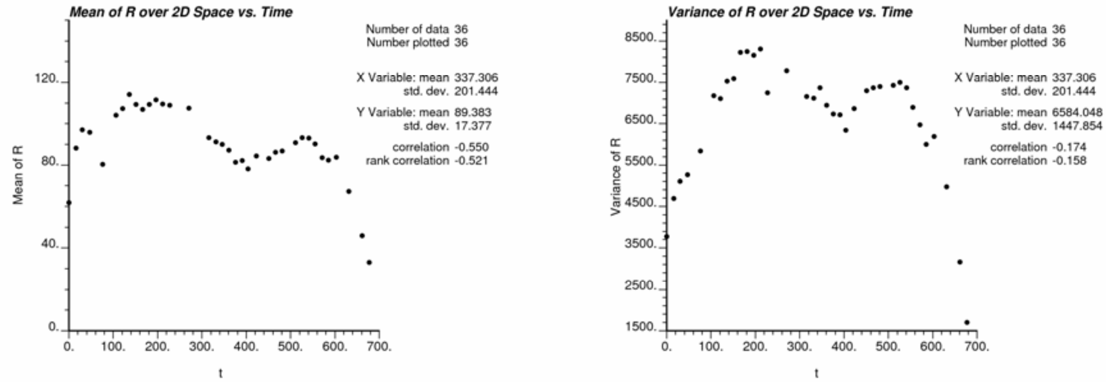

Figure 4: The scatter plots of mean of R over space versus time (left) and variance of R over space versus time.

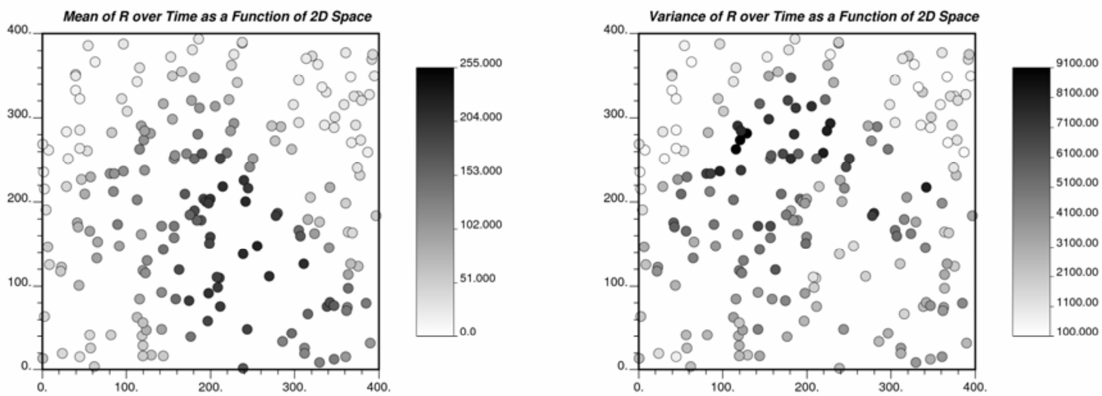

Figure 5: The 2D map of mean of R over time (left) and variance of R over time.

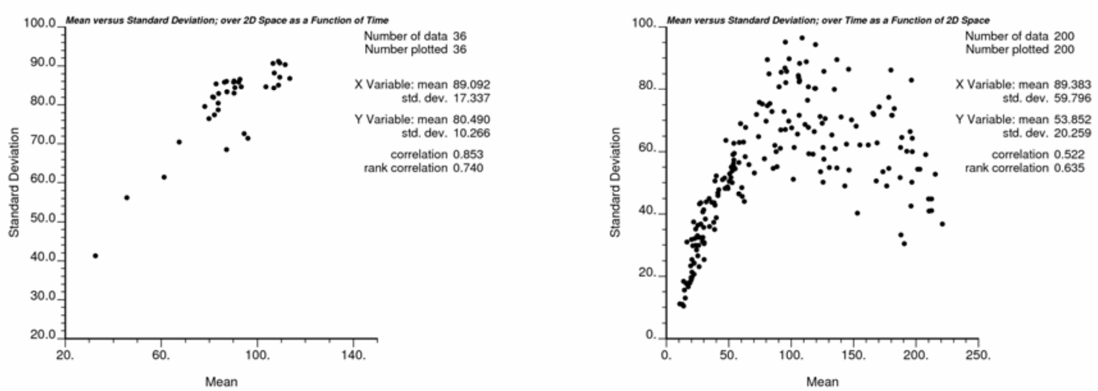

Figure 6: The scatter plots of mean versus standard deviation; over space as a function of time (left) and over time as a function of space (right).

The distribution of R samples for these two time steps are shown in Figure 7. To see better the change of the shape of the histogram, the difference of the mean and median (skewness) at each time step over space is plotted against time, it is shown in Figure 8.

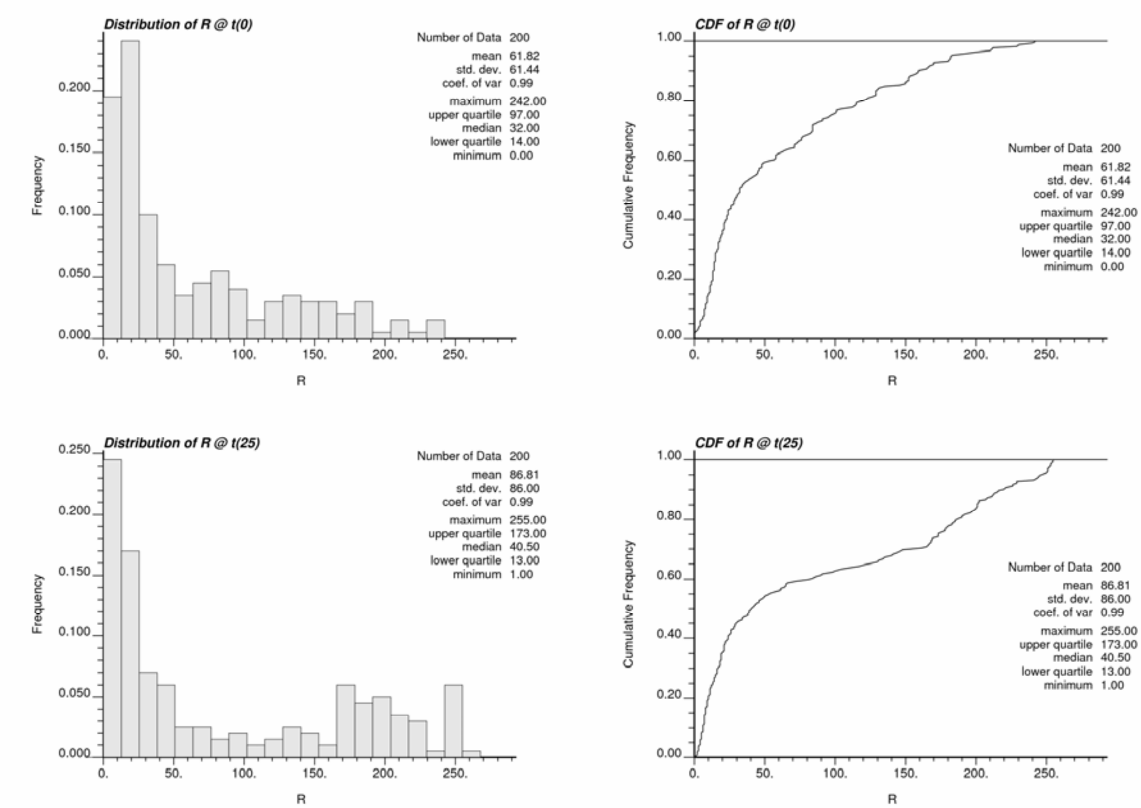

Figure 7: The histogram and cumulative histogram of R values at $t = t_0$ (top row) and at $t = t_{25}$ (bottom row).

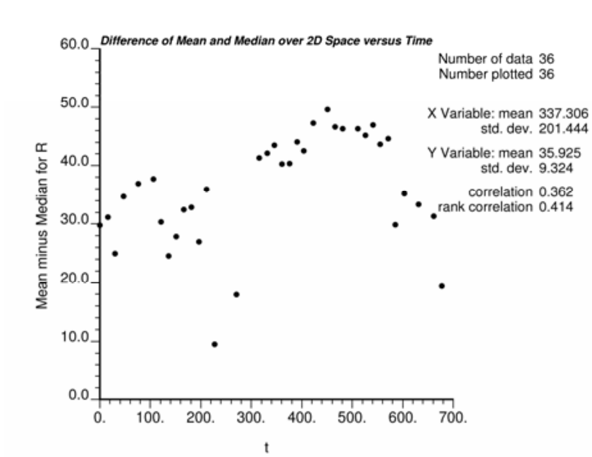

Figure 8: Mean minus median over space as a function of time which represents the changes of the shape of distribution as time increases.

The spatial normal scores variogram at two different time steps are shown in Figure 9. Figure 10 shows the temporal variogram at 2 different locations, the temporal variograms in this figure are calculated by using the normal score transformed data in 2-D space that already exist (normal score transform by using 200 value at each time step). The spatial variograms are expected to have a sill of 1.0 (based on transformation to standard normal at each time step); given that the same data was used to calculate the temporal variograms, it is expected that the temporal sill is not 1.0 since all 36 temporal samples available for any one well location are not ensured to be standard normal. Interestingly and not surprisingly, there is a very clear hole-effect phenomenon that is apparent in the temporal variogram despite the fact that it is more pronounced at some locations than at others.

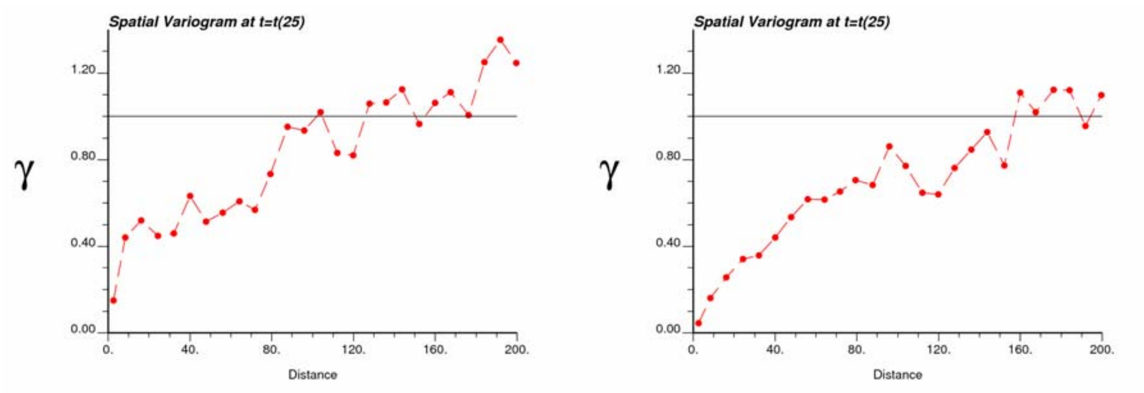

Figure 9: Spatial variogram in normal score units at $t = t_0$ (left) and at $t = t_{25}$ (right).

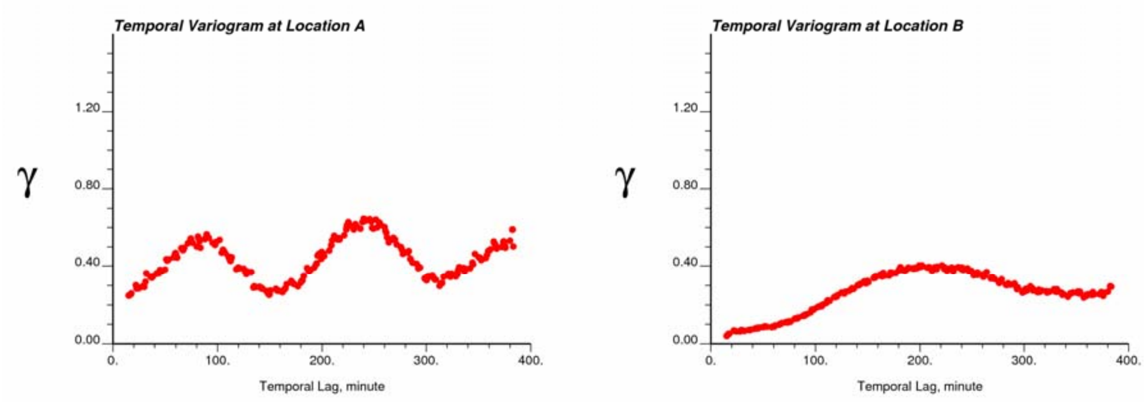

Figure 10: Temporal variograms at two different sampled locations by using the spatial normal score data.

The Synthetic Spatiotemporal Variogram Models

The first step in obtaining the spatiotemporal variogram model for the synthetic case is calculating the spatiotemporal variogram at both zero temporal lag distance, $\gamma_{ST}(\mathbf{h}_s, 0)$ and zero spatial lag distance, $\gamma_{ST}(0, \mathbf{h}_t)$.

The experimental spatiotemporal variogram can be calculated by using the following formula:

$$\gamma_{ST}(\mathbf{h}_s,\mathbf{h}_t)=\frac{1}{2N(\mathbf{h}_s,\mathbf{h}_t)}\sum_{N(\mathbf{h}_s,\mathbf{h}_t)}\left[z(\mathbf{u},t)-z(\mathbf{u}+\mathbf{h}_s,t+\mathbf{h}_t)\right]^2$$

where $N(\mathbf{h}_s,\mathbf{h}_t)$ is the number of pairs of sample points separated by spatial lag of $\mathbf{h}_s$ and temporal lag of $\mathbf{h}_t$. Since in our synthetic case the locations of the monitoring stations are fixed with respect to time therefore $N(\mathbf{h}_s,\mathbf{h}_t)$ is equal to the multiplication of $N_s(\mathbf{h}_s)$ and $N_t(\mathbf{h}_t)$, where $N_s(\mathbf{h}_s)$ is the number of pairs of sample points separated by spatial lag of $\mathbf{h}_s$ and $N_t(\mathbf{h}_t)$ is the number of pairs of sample points separated by temporal lag of $\mathbf{h}_t$. Therefore the above formula can be reduced to

$$\gamma_{ST}(\mathbf{h}_s,\mathbf{h}_t)=\frac{1}{2N_s(\mathbf{h}_s)N_t(\mathbf{h}_t)}\sum_{i=1}^{N_s(\mathbf{h}_s)}\sum_{j=1}^{N_t(\mathbf{h}_t)}\left[z(\mathbf{u}_i,t_j)-z(\mathbf{u}_i+\mathbf{h}_s,t_j+\mathbf{h}_t)\right]^2$$

At zero temporal lag it can be written in this way, we have

$$\gamma_{ST}(\mathbf{h}_s,0)=\frac{1}{N_t(0)}\sum_{j=1}^{N_t(0)}\left[\frac{1}{2N_s(\mathbf{h}_s)}\sum_{i=1}^{N_s(\mathbf{h}_s)}\left[z(\mathbf{u}_i,t_j)-z(\mathbf{u}_i+\mathbf{h}_s,t_j)\right]^2\right]$$

where $N_t(0)$ is equal to the number of time steps which is equal to n_t. Therefore $\gamma_{ST}(\mathbf{h}_s,0)$ is nothing more than the average of spatial variograms over time. At each time step the experimental spatial variogram is calculated at each spatial lag, $\mathbf{h}_s$, and at last we take an average of these experimental values, this average is the experimental spatiotemporal variogram value at the spatial lag, $\mathbf{h}_s$. In the synthetic case the number of time steps (n_t) is 36, therefore there are 36 spatial variograms.

A similar development exists for the spatiotemporal variogram at zero spatial lag, $\gamma_{ST}(0,\mathbf{h}_t)$, the formula for this case is

$$\gamma_{ST}(0,\mathbf{h}_t)=\frac{1}{N_s(0)}\sum_{i=1}^{N_s(0)}\left[\frac{1}{N_t(\mathbf{h}_t)}\sum_{j=1}^{N_t(\mathbf{h}_t)}\left[z(\mathbf{u}_i,t_j)-z(\mathbf{u}_i,t_j+\mathbf{h}_t)\right]^2\right]$$

where $N_s(0)$ is equal to the number of points in space, which is equal to 200 in our case.

Figure 11 shows the spatiotemporal variogram at zero temporal lag, $\gamma_{ST}(\mathbf{h}_s,0)$ and at zero spatial lag, $\gamma_{ST}(0,\mathbf{h}_t)$, which were calculated using the above formulas. The fitted models are also shown on this figure and are given below:

$$\gamma_{ST}(\mathbf{h}_s,0)=0.1+0.9\times Sph\left(\frac{\mathbf{h}_s}{150}\right)$$

$$\gamma_{ST}(0,\mathbf{h}_t)=0.08+0.132\times Exp\left(\frac{\mathbf{h}_t}{50}\right)+0.2537\times Sph\left(\frac{\mathbf{h}_t}{190}\right)$$

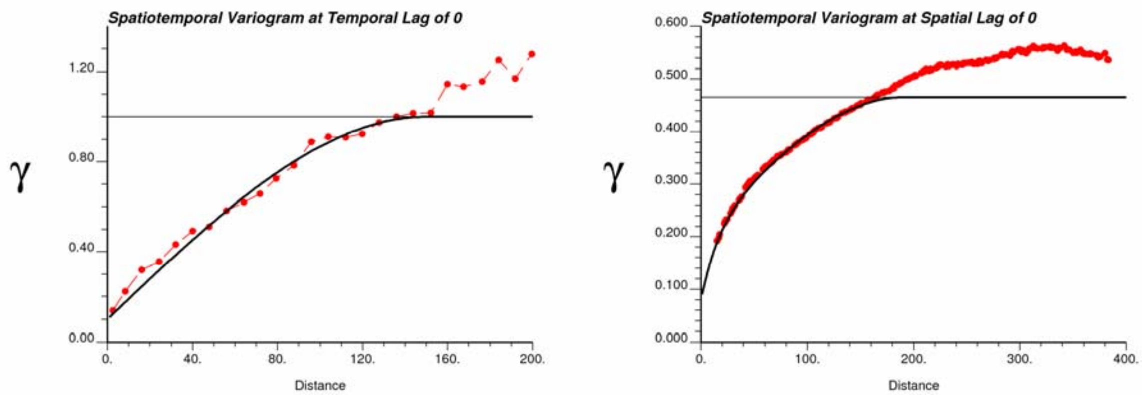

Figure 11: The spatiotemporal variogram at zero temporal lag (left), $\gamma_{ST}\left(\mathbf{h}_s,0\right)$, and the spatiotemporal variogram at zero spatial lag (right), $\gamma_{ST}\left(0,\mathbf{h}_t\right)$ along with the fitted models.

Among the separable models that were discussed, the sum model is the easiest one to use but it has some restrictions in data configuration of hard data that may cause the covariance matrix be noninvertible (Myers and Journel, 1990). The marginal spatiotemporal variograms in both metric models must have the same type of variogram structures and common variance contributions, only the range parameter can change. This results in a comparatively restrictive approach in modeling if we are to adopt such a model. Although the sum-metric model removes the restriction of the sum model due to possible data configuration issues, it still has the same restriction as the other metric models. The most useful (and easiest to fit) models among these separable models are product and product-sum model. The product model can be considered as a special case of product-sum model. An assumption of second order stationarity is essential here in order to compare the sum, metric and sum-metric models.

The experimental spatiotemporal variogram is shown in Figure 12; the spatiotemporal marginal variogram (spatial and temporal) can be seen easily from the 3D plot. Sum, product and product-sum models are used to model the experimental spatiotemporal variogram. Three sills can be observed in this plot, (1) the global sill (when both $\mathbf{h}_s$ and $\mathbf{h}_t$ approaches infinity), (2) the spatial sill (when $\mathbf{h}_s$ approaches infinity and $\mathbf{h}_t$ approaches zero) (3) the temporal sill (when $\mathbf{h}_s$ approaches zero and $\mathbf{h}_t$ approaches infinity).

Based on the sum, product and product-sum model the spatiotemporal variogram models can be obtained. These three models can be obtained by using two marginal variograms (spatial and temporal). Based on the sum model (Figure 13):

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma_{ST}\left(\mathbf{h}_s,0\right)+\gamma_{ST}\left(0,\mathbf{h}_t\right)$$

The product model (Figure 14):

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma_{ST}\left(\mathbf{h}_s,0\right)+\gamma_{ST}\left(0,\mathbf{h}_t\right)-1.0064\times\gamma_{ST}\left(\mathbf{h}_s,0\right)\times\gamma_{ST}\left(0,\mathbf{h}_t\right)$$

The product-sum model (Figure 15):

$$\gamma_{ST}\left(\mathbf{h}_s,\mathbf{h}_t\right)=\gamma_{ST}\left(\mathbf{h}_s,0\right)+\gamma_{ST}\left(0,\mathbf{h}_t\right)-1.0137\times\gamma_{ST}\left(\mathbf{h}_s,0\right)\times\gamma_{ST}\left(0,\mathbf{h}_t\right)$$

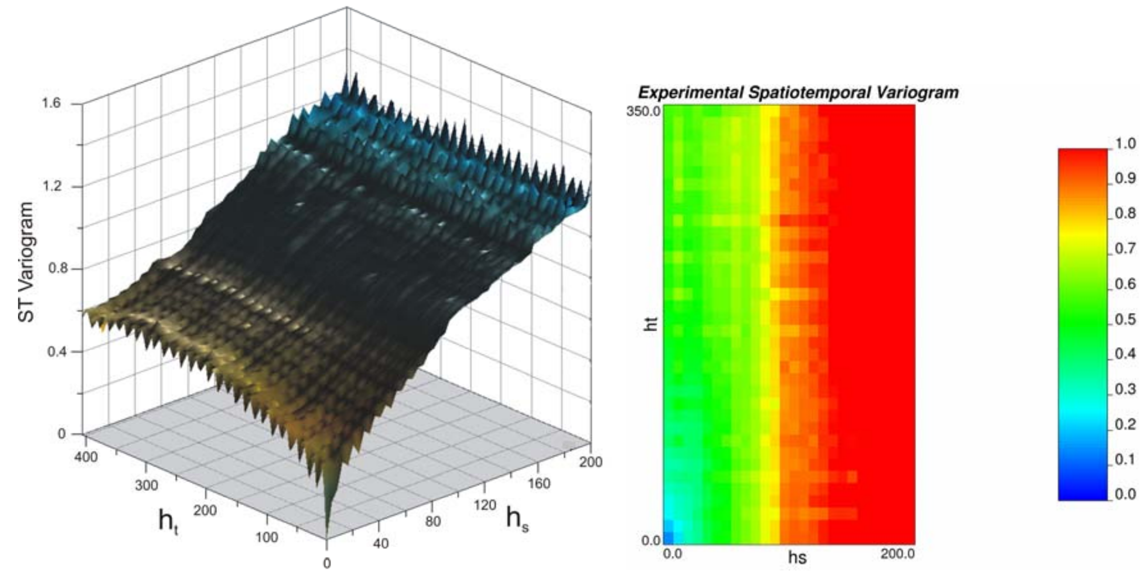

Figure 12: The experimental spatiotemporal variogram, $\hat{\gamma}_{ST}\left(\mathbf{h}_s, \mathbf{h}_t\right)$.

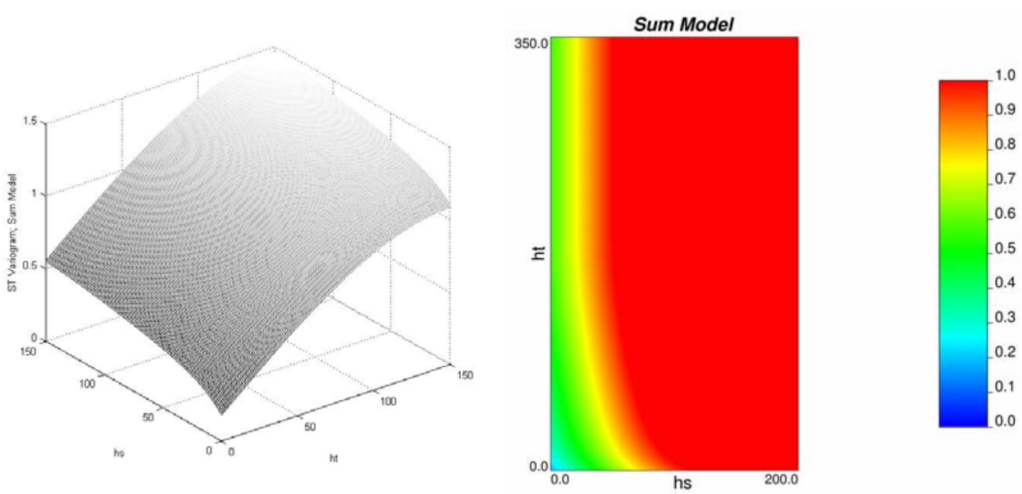

Figure 13: The synthetic spatiotemporal variogram by using the sum model.

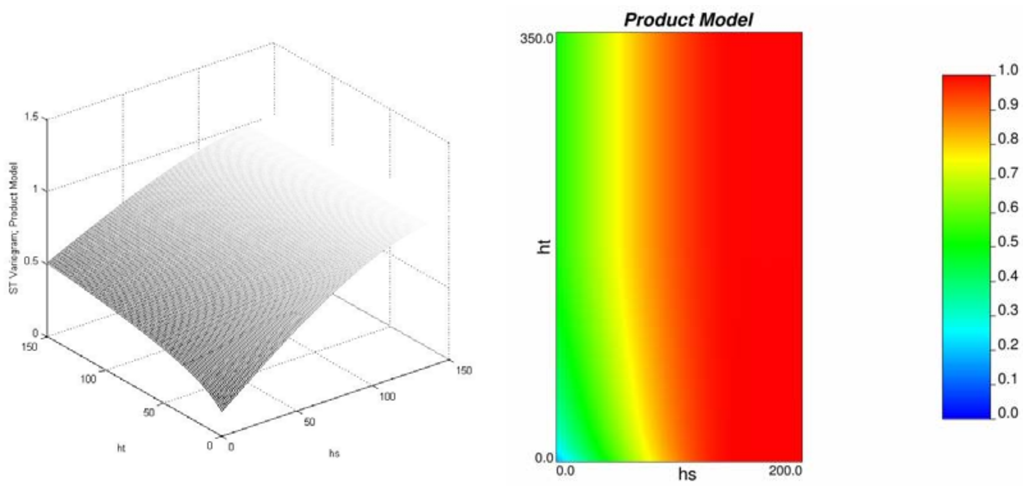

Figure 14: The synthetic spatiotemporal variogram by using the product model.

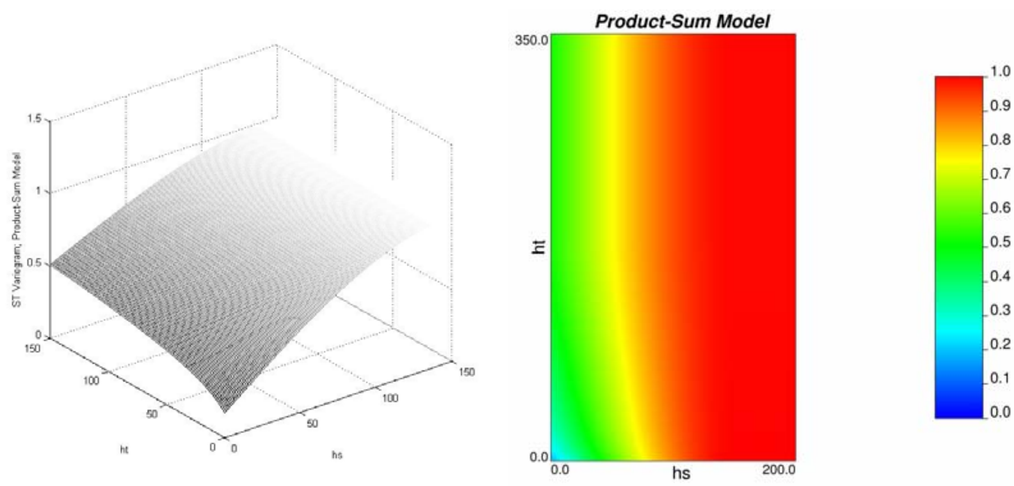

Figure 15: The synthetic spatiotemporal variogram by using the product-sum model.

Comparisons between the Fitted Models

It can be seen from Figures 13 to 15 that for our synthetic case, the spatiotemporal variogram for the product and product-sum model are approximately the same. The difference between the product and product-sum model is in the coefficient of the multiplication of two marginal variograms, for the product model the coefficient is equal to the inverse of the global spatiotemporal sill.

$$k = \frac{1}{C_{ST}\left(0,0\right)}$$

But for the product-sum model the coefficient is

$$k = \frac{C_S(0) + C_T(0) - C_{ST}(0,0)}{C_S(0) \times C_T(0)}$$

Another difference between the product and product-sum model is in the form of the covariance models, although the variogram models have the same structure, the covariance models do not. The difference between product and product sum model can be calculated as

$$d(\mathbf{h}_s, \mathbf{h}_t) = 0.0073 \times \gamma_{ST}(\mathbf{h}_s, 0) \times \gamma_{ST}(0, \mathbf{h}_t)$$

This difference is also illustrated in

Figure 16.

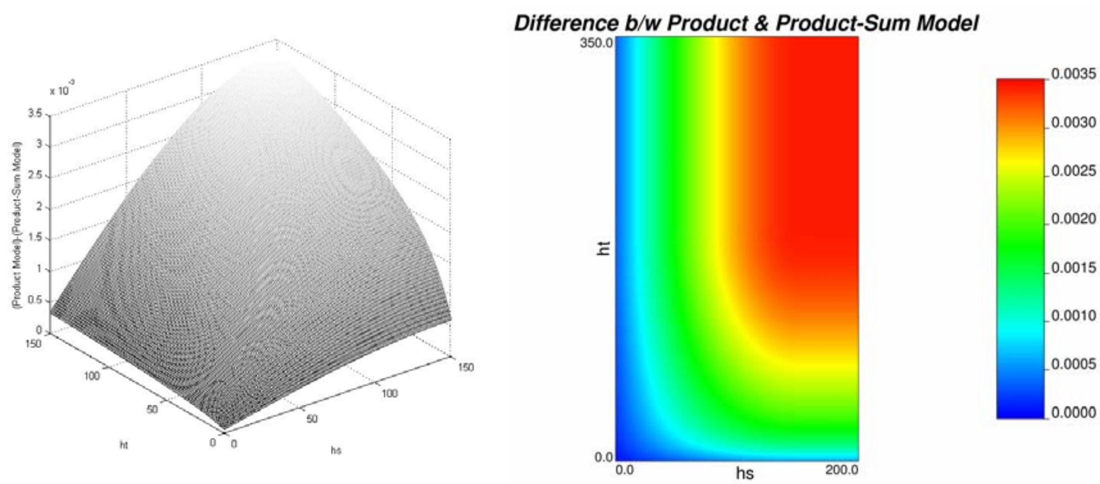

Figure 16: Difference between product and product-sum model

These three models can be compared to the experimental variogram by calculating an objective function, similar to that defined by Larrondo et al. (2003) for variogram fitting. Specifically, the objective function is

$$\text{Objective Function} = \sum_{ilag=1}^{n_{s,lag}} \sum_{jlag=1}^{n_{t,lag}} \left[\gamma(\mathbf{h}_{s,ilag}, \mathbf{h}_{t,jlag}) - \hat{\gamma}(\mathbf{h}_{s,ilag}, \mathbf{h}_{t,jlag}) \right]^2$$

where *ilag* is the index for each experimental spatial lag distance, *jlag* is the index for each experimental temporal lag distance, $n_{s,lag}$ is the number of spatial lag distances, $n_{t,lag}$ is the number of temporal lag distances, $\hat{\gamma}(\mathbf{h}_{s,ilag}, \mathbf{h}_{t,jlag})$ is the experimental spatiotemporal variogram and $\gamma(\mathbf{h}_{s,ilag}, \mathbf{h}_{t,jlag})$ is the fitted spatiotemporal variogram model which can be the sum, product or product-sum models, at each spatial $\mathbf{h}_{s,ilag}$ and temporal lag distance $\mathbf{h}_{t,ilag}$. Results from applying this objective function are provided in the table below.

Type of the Model	Objective Function
Sum	123.2238
Product	17.2218
Product-Sum Model	17.3516

Based on calculating the objective function for these three cases, we see that the product and product-sum models are much closer to the experimental variogram than the sum model. Although the product model yields the lowest objective function value,

Figure 16 shows that the difference between the product and product-sum models is in the order of 10^{-4} and can be considered insignificant. This is also consistent with the highly comparable objective function values for these models.

Final Remarks

Six different separable spatiotemporal models of regionalizations were discussed in this paper. A small comparative study was used to compare the sum, product and product-sum models. We could also consider nonseparable models to see better the interaction between time and space. Along with this step, kriging can also be considered for spatiotemporal processes to yield a set of smooth spatial models for different time steps. This gives a kind of interpolation / extrapolation in time. The case of extrapolation with respect to time is nothing more than forecasting which is critical in meteorology and environmental sciences. Another area of research naturally extends from estimation and that is to perform geostatistical simulation for the spatiotemporal data sets.

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