Determination of Locally Varying Directions through Mass Moment of Inertia Tensor

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The direction and magnitude of geological continuity is often revealed by geological interpretation. These details of geological continuity/variability can be important for resource/reserve calculations. The connectivity of high and low values are critical for prediction. Geological interpretation is essential, but is often limited to large scale features and generalizations. An automatic approach is required for local refinement and repeatability. The variogram map (or volume in 3-D) is commonly used to assist in the determination of variogram directions. The correlation map (variance minus the variogram map) could be interpreted as "mass" and automatically fit by a tensor. The principal directions of this tensor could be used in subsequent modeling. A logical extension of this idea is to calculate these maps locally and fit locally varying directions of continuity. These ideas are developed in this paper.

Introduction

The geological continuity is a key factor for prediction of fluid flow in reservoir. The geological continuity and variogram continuity are direction dependent (Deutsch, 2002). Generally, the directions of continuity in variables are determined prior to geostatistical modeling. The fairly standard approach is to plot the variogram maps and detect the maximum and minimum continuity direction based on the range of variogram. The geological information and interpretations are another source to detect the continuity direction in lithofacies. This process is subjective and requires the user judgment. In the cases that we are dealing with large number of variables and geological facies, applying this method could be frustrating. Using tensor of moment of inertia can be helpful.

Tensors are generally used to show the variation of an isotropic variable in different directions. Tensor is a 2 by 2 matrix in 2D or a 3 by 3 in 3D cases. The main diagonal component of each tensor indicates the value of anisotropic variable in the principal coordinate axes (e.g. *X*, *Y* and *Z* Cartesian coordinates) while the off-diagonal terms shows the value respect to other arbitrary axes.

Given the components of the symmetric tensor in a coordinate system, we can find all components in any other coordinate system. Consider an anisotropic property, *A*, in an anisotropic medium. The value of variable *A* based on the coordinate axes of x, y and z is shown with the following symmetric tensor:

$$A = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix}$$

in which $A_{xy} = A_{yx}$, $A_{xz} = A_{zx}$ and $A_{yz} = A_{zy}$.

For a new coordinate system (*X*', *Y*' and *Z*') which is θ° apart from the original axes the relationship can be derived. For the two dimensional case, these relationships are:

$$A_{x'x'} = \frac{A_{xx} + A_{yy}}{2} + \frac{A_{xx} - A_{yy}}{2}\cos 2\theta - A_{xy}\sin 2\theta \qquad (1)$$

$$A_{y'y'} = \frac{A_{xx} + A_{yy}}{2} - \frac{A_{xx} - A_{yy}}{2} \cos 2\theta + A_{xy} \sin 2\theta \tag{2}$$

$$A_{x'y'} = \frac{A_{xx} - A_{yy}}{2} \sin 2\theta + A_{xy} \cos 2\theta \tag{3}$$

where $A_{x'x'}, A_{y'y'}$ and $A_{x'y'}$ are the component of tensor in new coordinate system.

The equation (1) and (3) are the parametric equation of a circle. This circle is called Mohr's circle. Figure 1 shows an example of Mohr's circle. The Mohr's circle shows the variation of tensor components when the angle θ is changing. According to this circle we can find an angle θ such that the cross terms are became zero and the tensor will be diagonal ($A_{x'y'} = A_{y'x'} = 0$). The points B and D on the circle show two cases where the tensor is diagonal and $A_{x'x'}$ is minimum and maximum respectively. Those directions that have these properties are called the *principal directions* of tensor. The angle θ can be defined by the following formula:

$$\tan 2\theta = -\frac{2A_{xy}}{A_{xx} - A_{yy}} \tag{4}$$

The equation defines two values of 2θ which are 180° apart and thus two values of θ which are 90° apart. In order to define which angle is related to $A_{\max}$ and which one is related to the $A_{\min}$, we can substitute both values of θ into equation (4) and define the maximum value of *A*.

For three dimensional tensors, the principal directions can be defined by eigenvalue and eigenvector decomposition. This needs to solve the following equation:

$$\begin{bmatrix} A_{xx} - \lambda & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} - \lambda & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} - \lambda \end{bmatrix} . \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = 0 \tag{5}$$

where v_x, v_y and v_z are the components of the eigenvector of principal direction and λ is the corresponding eigenvalue.

The three eigenvalues of equation (5) are the roots of the following cubic equation:

$$\begin{vmatrix} A_{xx} - \lambda & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} - \lambda & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} - \lambda \end{vmatrix} = \lambda^3 - a\lambda^2 + b\lambda - c = 0 \tag{6}$$

where the coefficients are:

$$a = A_{xx} + A_{yy} + A_{zz}$$
$$b = A_{xx}.A_{yy} + A_{yy}.A_{zz} + A_{xx}.A_{zz} - A_{xy}^2 - A_{yz}^2 - A_{xz}^2$$
$$c = A_{xx}.A_{yy}.A_{zz} + 2A_{xy}.A_{yz}.A_{xz} - A_{xx}.A_{yz}^2 - A_{yy}.A_{xz}^2 - A_{zz}.A_{xy}^2$$

Once the eigenvalues are calculated, the eigenvectors are determined by substituting them into equation (5). Finally we have three eigenvalues and three corresponding eigenvectors which are representing the principal directions. It has been shown (Beer et. al., 1988) that the eigenvalues are the values of the variable

A in the principal directions. Based on this statement the major principal direction is related to the one with the greater eigenvalue.

Moment of Inertia

The moment of inertia of a body is related to the distribution of the mass throughout the body and quantifies the rotational inertia of a rigid body.

$$I = \int_V m.r^2 dm \tag{7}$$

where m is mass and r is the perpendicular distance from the axis of rotation.

Generally, there are two forms of moment of inertia; scalar form which is used when the axis of rotation is known and the tensor form that summarizes all moment of inertia for different axes of rotation with one quantity.

For a rigid body consisting of N point masses m_i the moment of inertia tensor are defined as follows:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{8}$$

$$I_{xx} = \sum_{i=1}^{N} m_i \left(y_i^2 + z_i^2 \right) \quad , \quad I_{xy} = I_{yx} = \sum_{i=1}^{N} m_i x_i y_i$$

$$I_{yy} = \sum_{i=1}^{N} m_i \left(x_i^2 + z_i^2 \right) \quad , \quad I_{xz} = I_{zx} = \sum_{i=1}^{N} m_i x_i z_i$$

$$I_{zz} = \sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2 \right) \quad , \quad I_{yz} = I_{yz} = \sum_{i=1}^{N} m_i y_i z_i$$

where x_i, y_i and z_i are the distances of point i from the coordinate axes. Here the physical meaning of I_{xx} is the moment of inertia around the *x*-axis when the objects are rotated around the x-axis and I_{xy} is the moment of inertia around the *y*-axis when the objects are rotated around the *x*-axis.

Since the moment of inertia is an anisotropic quantity and presented as a tensor, the principal directions can be determined with the same approach as discussed in above. Here the principal directions of moment of inertia show the directions in which the rigid body is more closely distributed or less distributed. In this case the major direction is the one which related to the smaller moment of inertia. This interesting property leads us to determine the major and minor direction of continuity in the geological setting or define the principal direction of an unstructured grid block.

Geological Models

The correlation indicates the strength and direction of a linear relationship between two random variables which are apart by vector **h**.

$$\rho_{X,Y} = \frac{Cov\{X,Y\}}{\sigma_X.\sigma_Y} = \frac{E\{(X-\mu_X)(Y-\mu_Y)\}}{\sigma_X.\sigma_Y} \tag{9}$$

The correlation varies between -1 to 1. Correlation of 1 shows the increasing linear relationship and -1 indicates the decreasing linear relationship. The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables. If the variables are independent the correlation is zero.

The correlation map shows the calculated correlation of a data set for different lag **h** and directions. Considering the correlation as a mass quantity, the correlation map can be considered as a rigid body and we can find the maximum and minimum direction of continuity analogous to method used to calculate the principal axes for moment of inertia.

Given any continues geological variable model or categorical facies model, the following steps leads to find the direction of continuity:

1. Generate the correlation map for continues model. For categorical model, each code can be considered as a mass quantity.
2. Calculate the moment of inertia tensor for the point masses according to the x, y and z axes pass through the center of mass.
3. Determine the principal direction of continuity using the method discussed above.

We verify the method with the following examples.

Examples

Two models are considered and the principal directions are determined. For the first example a continues model is generated with the unconditional Sequential Gaussian Simulation (SGSIM) with the azimuth angle of 60° and dip angle of 0°. The correlation map is calculated and plotted for the range of 1/3 of the field size. Figure 2 shows the model and the correlation map.

The moment of inertia tensor is calculated and the principal directions are defined by the equation (8).

$$\begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} = \begin{bmatrix} 4514.33 & 4712.47 \\ 4712.47 & 10408.78 \end{bmatrix}$$

$$\tan 2\theta = -\frac{2I_{xy}}{\left(I_x - I_y\right)} \qquad \Rightarrow \qquad \begin{cases} \theta_1 = 61.01^\circ \\ \theta_2 = -28.99^\circ \end{cases}$$

The major direction is defined by substituting θ_1 and θ_2 into the equation (4). The angle with the smaller moment of inertia is the major direction ($\theta_1 = 61.01^\circ$).

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad \Rightarrow \qquad \begin{cases} I_{x'}\left(\theta_1\right) = 5028.69 \\ I_{x'}\left(\theta_2\right) = 9894.42 \end{cases}$$

For the second example a single ellipse is generated with the azimuth angle of 20° and the dip angle of 0°. The girds inside the ellipse are coded as 1 and the outside as code 0 (Figure 3). The methodology is applied on this model and the inertia tensor is calculated.

$$\begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} = \begin{bmatrix} 114976.0 & 53370.0 \\ 53370.0 & 75764 \end{bmatrix}$$

The angles are calculated and the major direction is determined with the same method as last example($\theta_1 = -69.82^\circ$, $\theta_2 = 21.18^\circ$). The azimuth angle is reproduced with error of $\%5.8$.

Locally Varying Angles

The anisotropy in the geological setting is varying from one point to another due to the heterogeneity. In some complex geological setting such as Fluvial setting which contains channel features, the continuity direction significantly changes from one region to another. Geostatistical modeling techniques such as sequential Gaussian simulation or sequential indicator simulations with a global variogram orientation and search parameters can not reproduce this local changes in continuity direction. Some attempts have been done to change the simulation algorithms in order to consider the locally varying angles (Leuangthong et. al., 2006).

In the cases where a representing training images or any other geological model is available, the methodology discussed in this paper can be applied locally to determine the local varying continuity direction. This can be more explained with the following example.

Example

A 2D training image is considered. The model has 256 grids in both x and y direction. There are three facies in the model. In order to locally determine the continuity direction a window is considered which contains 16 fine grid cells in x and y direction. Figure 4 shows the training image and the windows which are considered.

A constant mass is assigned to facies type 2 which represents the channels. The mass of other facies are set to 0. The moment of inertia is calculated for each window and the results are plotted in Figure 5 .

Unstructured Grid Element

The orientation of unstructured grid blocks is important factor in calculating the upscaled permeability tensor. The upscaling technique discussed in chapter three is based on the fine grid cells permeabilities which are considered for the volume averaging. It is obvious that for a single unstructured grid block imposed on a specific fine scale permeability model, the upscaled permeability tensor is related to the orientation of unstructured grid block.

The orientation of unstructured blocks can be determined with the same method as discussed in previous section. Each unstructured block is refined locally with the fine Cartesian grid cells and then the inside grids are coded as 1 while the outside cells are defined as code 0. Then the moment of inertia tensor is calculated and the major direction is defined. This is more investigated by the following example.

A 2D unstructured grid model is considered. There are 25 grid blocks in this model. Figure 6 shows the unstructured grid model. The methodology is applied for each unstructured grid block and the orientation is determined. Figure 7 shows the orientation of each grid blocks in the unstructured model.

References

Beer, Ferdinand P. and E. Russell Johnston, Jr. , 1988, *Vector Mechanics for Engineers: Statistics and Dynamics* (5^{th} ed). McGraw-Hill, p.1280.

Deutsch, C.V., 2002, Geostatistical Reservoir Modeling, New York, Oxford University Press, p.376.

Leuangthong, O., Prins, C., V. Deutsch, 2006, SGSIM_LVA: Gaussian Simulation with Locally varying Angles, Centre for Computational Geostatistics, Report 8, 408-1,408-7.

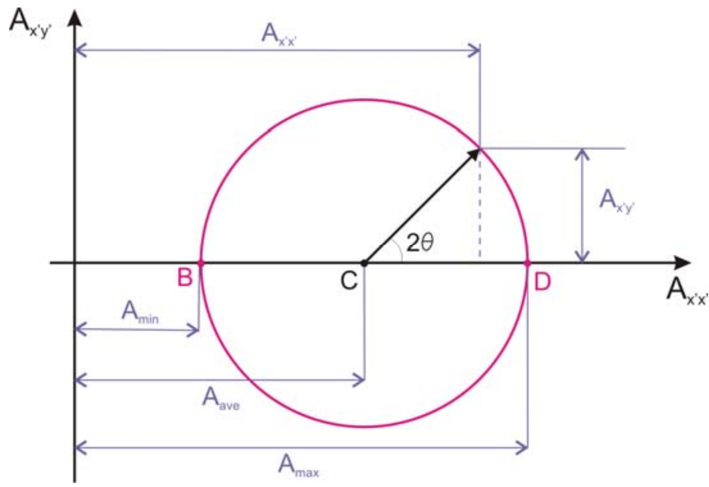

Figure 1: Mohr's Circle.

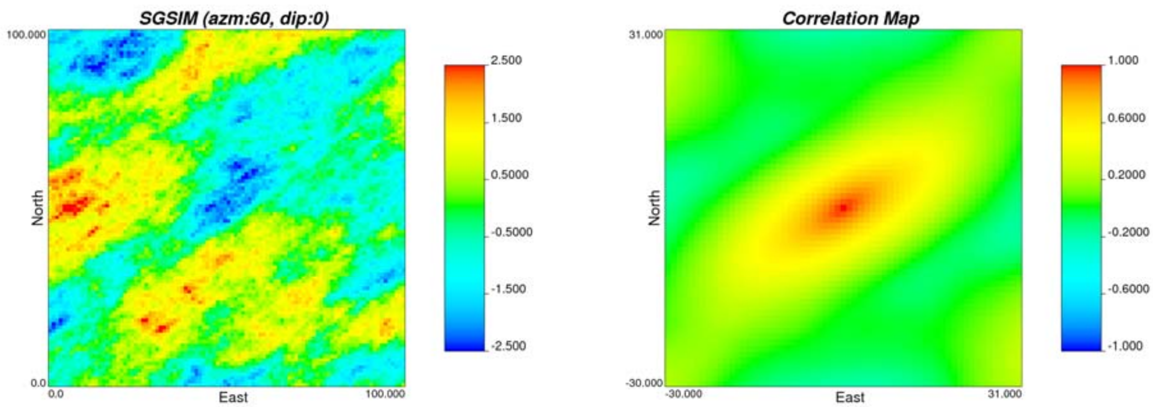

Figure 2: A synthetic continues model generated with azimuth of 60 (left) and the calculated correlation map (right).

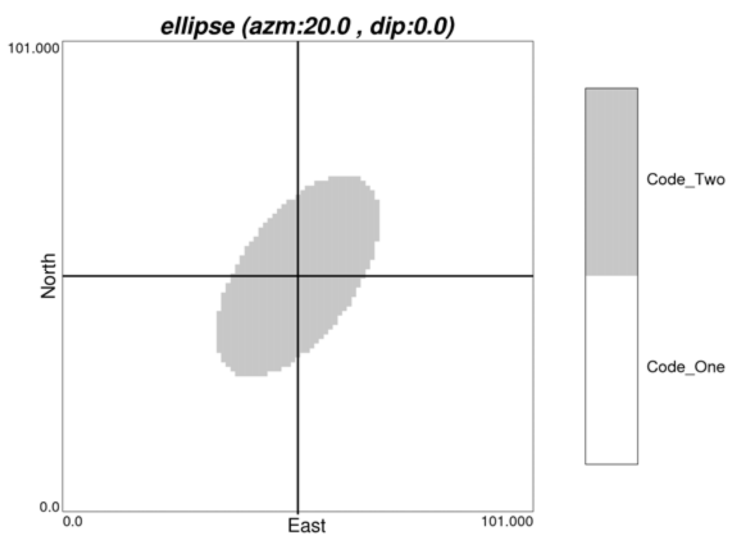

Figure 3: A simple categorical example generated with azimuth angle of 20.

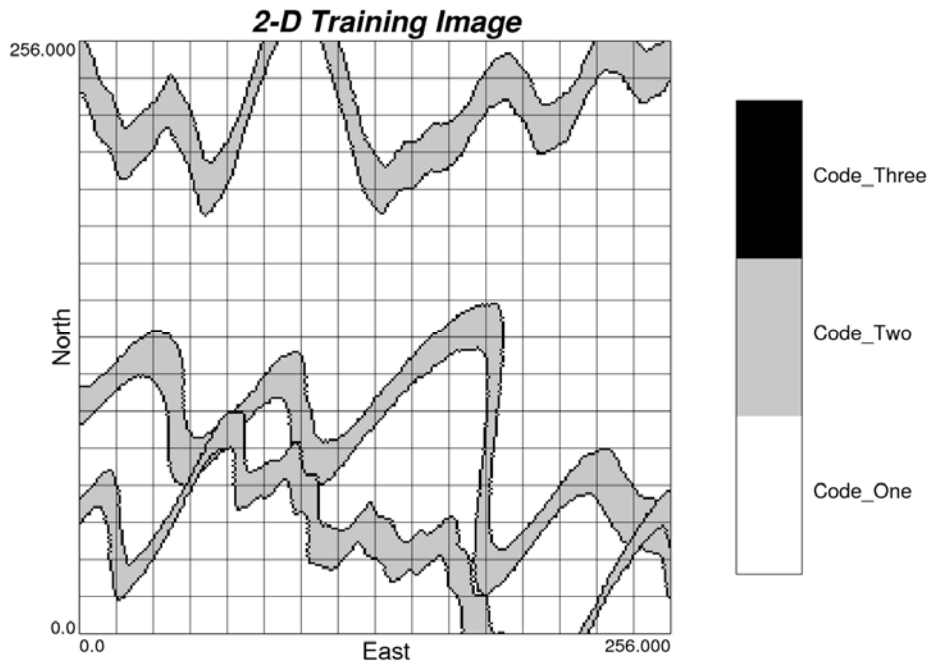

Figure 4: The training image considered for determination of locally varying angle. The squares show the windows.

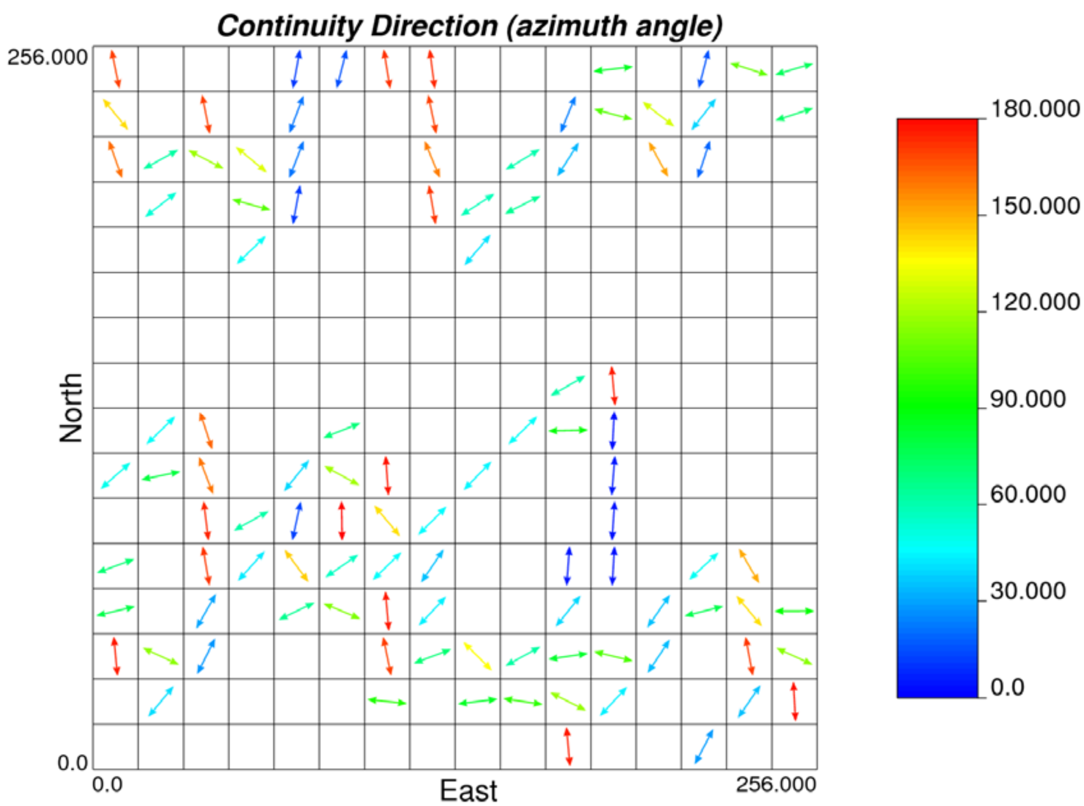

Figure 5: Locally varying angles. The arrows show the azimuth angle of continuity direction for each window.

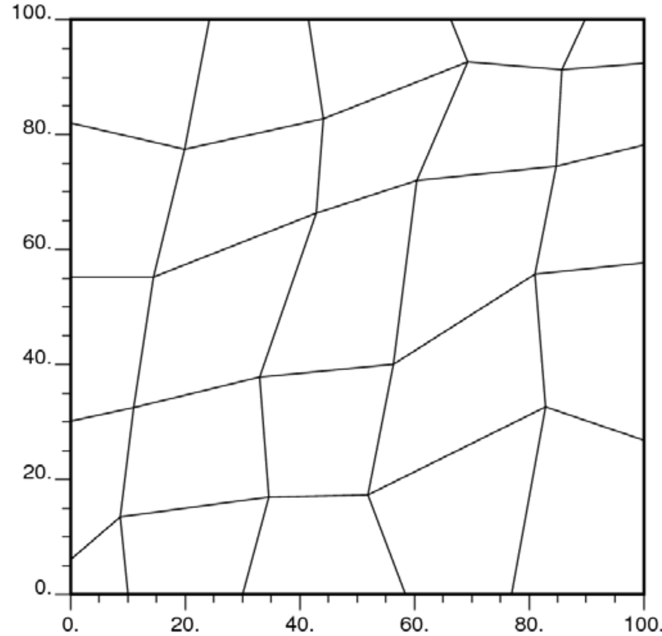

Figure 6: The unstructured grid model used for this example.

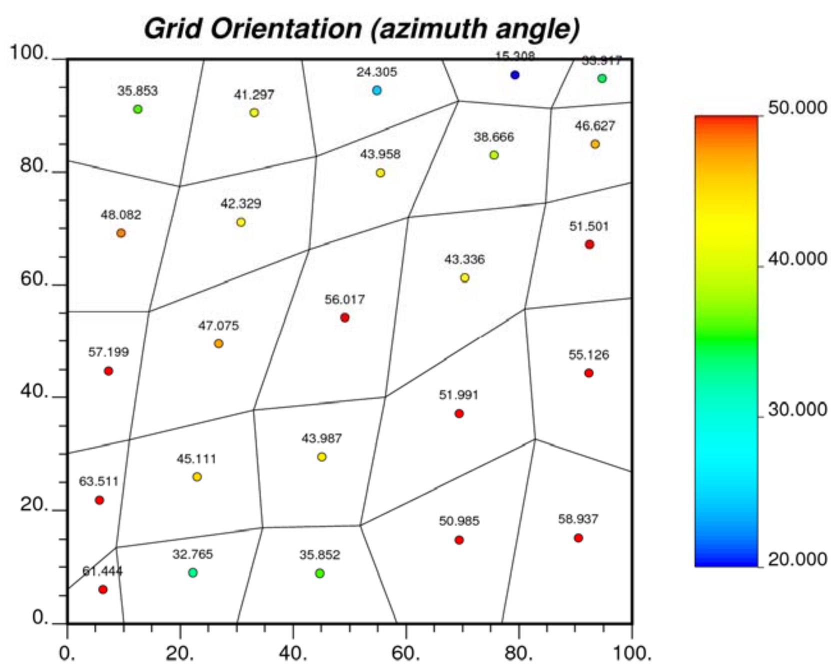

Figure 7: The unstructured grid orientation. The points show the centre of mass of each grid and the numbers show the azimuth angle of major direction.