Generalized Linear Distributions to Account for Parameter Uncertainty in Sequential Gaussian Simulation

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Geological sites are heterogeneous and uncertain. Multiple realizations are an important aspect of uncertainty; however, parameter uncertainty, that is, uncertainty in the histogram is even more important for uncertainty in resources and reserves. Accounting for parameter uncertainty in geostatistical simulation is a longstanding problem. Targeting specific quantiles such as P10, P50 and P90 realizations is a related challenge. This paper presents a solution to these problems. A methodology is presented to simulate realizations of continuous variables with specified position in the range of global uncertainty. The key to the methodology is the use of the generalized linear distribution in place of the uniform distribution for simulation. The theoretical validity of this method is established, implementation details are discussed and examples are presented. This has a wide range of applicability in modern geostatistical reservoir modeling where global uncertainty is an important goal.

Introduction

Creating P10, P50 and P90 geostatistical reservoir models is an important task for flow simulation, risk analysis, reservoir forecasting and management. A base case model is always required. The 80% probability interval is common in the earth sciences. Higher probability intervals are often so large that they are difficult to use in risk qualified decision making.

There are some statistical methods to establish P10 and P90 reserve figures. The conventional approaches to estimate the reserves are divided into deterministic and probabilistic methods. The deterministic approach consists of volumetric, material balance and decline curve analysis and they use a single value for each parameter for estimating the reserves, there are no P10, P50 and P90 values in this method. The probabilistic approach uses a full range of values for each parameter in the reserve calculation. For example, the volumetric method could use a distribution of values for porosity, initial water saturation, formation volume factor and so on to get a range of values for the reserve. For the purpose of reserve estimation, National Instrument 51-101 (NI 51-101) defines P10, P50 and P90 (Robinson et al, 2004). P90 refers to proved reserves, P50 refers to proved and probable reserves and finally P10 refers to proved, probable and possible reserves. Based on NI 51-101 definition, P90 is less than P50, and P50 is less than P10. In this paper, P10 refers to a *p*-value of 0.9 and P90 refers to a *p*-value of 0.1 (the *p*-values in this paper are defined base on the statistical definition of cumulative distribution function). The problem with conventional statistical methods is that there are no specific realizations. It is not possible to run a flow simulator and assess the dynamic performance of the models under different conditions. It is highly desirable to have specific realizations that approximately represent the 80% probability interval.

The traditional geostatistical approach to finding P10 and P90 models is based on ranking procedures. Multiple realizations (often 100) are generated, and then some quick-to-calculate static reservoir attribute such as connected pore volume is chosen to rank the realizations. Realizations with specific position in the distribution of the static response are selected. The biggest weakness of this approach is that parameter uncertainty is often ignored. Parameter uncertainty is probably the most important of global uncertainty.

Parameter uncertainty can be calculated with a variety of techniques including the spatial bootstrap and the conditional finite domain (CFD) method (Babak et al, 2007-1, 2007-2). The challenge is to transfer this uncertainty into the geostatistical realizations. One approach would be to use different reference distributions; however, this is not fully implemented in most software and conditioning causes the resulting distribution to be close to the data distribution in any case.

In practice, multiple realizations are created without accounting for parameter uncertainty. These realizations can be ranked; however, they are all very similar because major differences due to parameter uncertainty are ignored. This is a significant problem when the reservoir is large relative to the variogram range. Fluctuations above the mean average out with fluctuations below the mean and all realizations are very similar.

A general concern with any realization that is claimed to be a specific P value (e.g., P10, P50 or P90) is that it is not the same P value at all locations. A P10 realization may have P90 values at some locations, P50 at others and so on. The P value of a realization must be considered as a global parameter with little local meaning.

The technique proposed here is a modification to the popular Sequential Gaussian Simulation (SGS) algorithm. SGS draws realizations from a multivariate Gaussian distribution based on recursively decomposing the multivariate distribution by Bayes Law. The conditional distributions at each step in SGS are based on the normal equations (also known as simple kriging). Realizations are drawn from the conditional distributions by Monte Carlo simulation from a uniform distribution of probability. The simulated values are assigned to grid nodes and used to condition subsequent simulated values. The use of the uniform distribution leads to realizations from the multivariate Gaussian distribution that reproduces the target global mean and variance.

The central ideal proposed in this paper is to use a non-uniform distribution for the Monte Carlo simulation, see Figure 1. Uniform probabilities lead to a P50 realization. A 10% high realization would be generated by using random numbers preferentially from values nearer to 1 (the red dashed curve). A 10% low realization would use random number preferentially from values nearer to 0 (the green long-dashed curve).

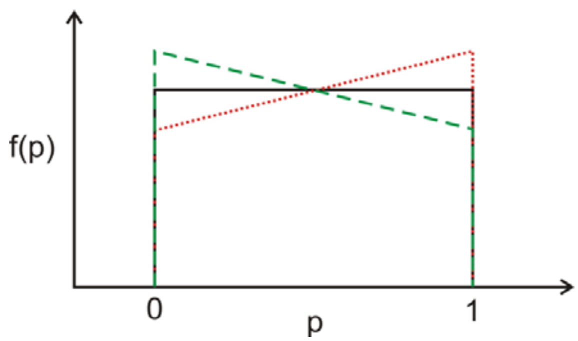

Figure 1: Non-uniform versus uniform distribution used for the Monte Carlo simulation; uniform distribution (black solid line) allows having a moderate combination of low and high values, the red dashed curve should be used to have a model with preferentially high values, the green long-dashed curve should be used to have a model with preferentially low values

This seems like an ad-hoc engineering approach with no statistical basis; however, this paper demonstrates the theoretical validity of this approach. The slope is analytically linked to the mean of the resulting distribution regardless of the distribution shape. This new family of linear distributions is referred to as a generalized linear distribution (GLD). This new linear family of distribution allows us to specify and achieve a target global mean with a known probability. There is no cumbersome pre- or post-processing. The only things that are needed to characterize the linear distribution are the original distribution of the data, the standard deviation of the global mean (which can be calculated by using bootstrap, spatial bootstrap, etc) and the *p*-value which we are interested in.

Standard Generalized Linear Distribution

The Generalized Liner Distribution (GLD) is a three-parameter (a, b, η) family of continuous probability distributions. a and b are the minimum and maximum values (these two parameters can be any value) and η is the slope or shape parameter (which should be between -1 and 1). The distribution can be abbreviated as $L(a, b, \eta)$. The standard GLD happens when a and b are 0 and 1 respectively. The probability density function (PDF), the cumulative distribution function (CDF) and the quantile function of the GLD are summarized below. If $\eta = 0$ the standard GLD returns to uniform distribution, $U(0,1)$.

PDF:

$$f\left(x;a,b,\eta\right)=\begin{cases}\dfrac{2\eta x+\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]}{\left(b-a\right)^{2}} & for \quad a\le x\le b\\ 0 \quad otherwise & \end{cases} \quad \text{..(1)}$$

CDF:

$$F\left(x;a,b,\eta\right)=\begin{cases}0 \quad for \quad x<a & \\ \dfrac{\eta\left(x^{2}-a^{2}\right)+\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]\left(x-a\right)}{\left(b-a\right)^{2}} & for \quad a\le x\le b\\ 1 \quad for \quad x>b & \end{cases} \quad \text{..(2)}$$

Quantile function:

$$q\left(p\right)=F^{-1}\left(p\right)=\left[\frac{\left(1+\eta\right)a-\left(1-\eta\right)b}{2\eta}\right]+\left(\frac{b-a}{2\eta}\right)\sqrt{\left(1-\eta\right)^{2}+4\eta p} \quad \text{..(3)}$$

Figure 2 shows the CDF and PDF of GLD. The characteristics of generalized linear distribution are discussed in the Appendix.

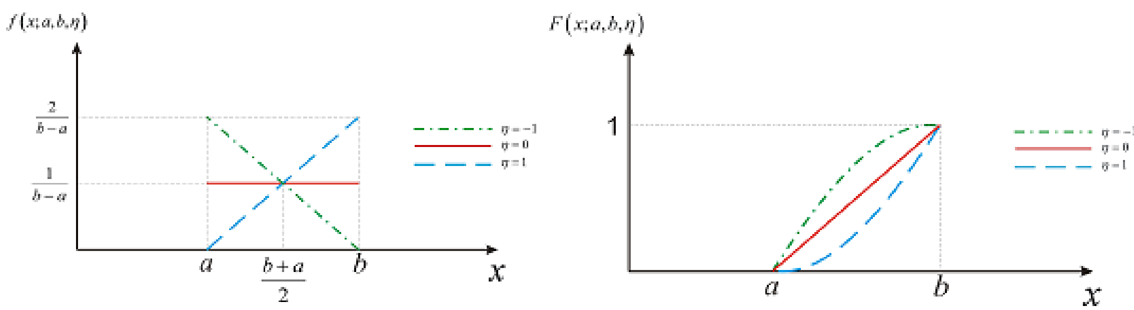

Figure 2: PDF (left) and CDF (right) of GLD for 3 different cases ($\eta=-1,0,1$)

Distribution of the global mean

To target the desired global mean with any probability which follows the Gaussian distribution using standard GLD we need to calibrate η for any p-value that we are interested in. Assume that m is the global mean; y_m is the global mean which follows the Gaussian distribution with mean of m and the standard deviation of σ_m. σ_m can be calculated by using bootstrap, spatial bootstrap, etc. It is also equal to $\sigma/\sqrt{N}$, where σ is the global standard deviation and N is the number of independent data.

Relationship between y_m, η and p

To derive the relationship between η and p, we need the quantile function of standard GLD, $q(p)$, and the original CDF of the data, $F(y)$. By using below equalities the relationship between y_m , η and p can be derived:

$$\left.\begin{aligned}q\left(p\right)&=F\left(y\right)\\ y_{m}&=\int_{0}^{1}y\cdot dp\end{aligned}\right\}\Rightarrow\quad \eta=\frac{y_{m}-m}{I} \quad \text{..(4)}$$

Where I can be calculated as:

$$I = E\left\{y \cdot \left[2F\left(y\right)-1\right]\right\} = \int_{-\infty}^{+\infty} y \cdot \left[2F\left(y\right)-1\right] \cdot f\left(y\right) \cdot dy \quad \text{.......(5)}$$

Where $f(y)$ and $F(y)$ are the global PDF and CDF of the data.

Since y_m follows the Gaussian distribution with mean of m and standard deviation of σ_m therefore the below relationship is true:

$$y_m = m + \sigma_m \cdot G^{-1}\left(p\right) \quad \text{.......(6)}$$

Where $G^{-1}(p)$ is the inverse of the standard normal cumulative distribution function. Inserting above relation in equation (4) yields to:

$$\eta = \frac{y_m - m}{I} = \left(\frac{\sigma_m}{I}\right) \cdot G^{-1}\left(p\right) \quad \text{.......(7)}$$

The value of σ_m/I ensures that η to be in the range of -1 to 1. Therefore we have a range for p-value of interest. Since σ_m is relatively small number (because of the narrow Gaussian distribution that we have for global mean) therefore p value approximately covers the whole range of 0 to 1.

$$-1 \le \left(\frac{\sigma_m}{I}\right) \cdot G^{-1}\left(p\right) \le 1 \quad \Rightarrow \quad G\left(\frac{-I}{\sigma_m}\right) \le p \le G\left(\frac{I}{\sigma_m}\right) \quad \text{.......(8)}$$

Where $G(p)$ is the standard normal cumulative distribution function. Equation (7) shows that η follows the Gaussian distribution with mean of zero and the standard deviation of σ_m/I. When the original distribution does not follow any specific parametric distribution, a good approximation of the original CDF would be:

$$\hat{F}\left(y\right) = \sum_{i=1}^{n} wt_i \cdot \Delta\left(y - y^{(i)}\right) \quad \text{.......(9)}$$

Where the sum of the weights is equal to one and $\Delta(y - y^{(i)})$ is the CDF of the Dirac distribution:

$$\Delta\left(y - y^{(i)}\right) = \begin{cases} 0 & if \quad y < y^{(i)} \\ 1 & if \quad y \ge y^{(i)} \end{cases} \quad \text{.......(10)}$$

$y^{(i)}$ is the i^{th} original data in the ascending order. Using Dirac distribution the equation for I would be:

$$I = \sum_{i=1}^{n} \sum_{j=1}^{n} wt_i \cdot wt_j \cdot \left[2\Delta\left(i-j\right)-1\right] \cdot y^{(i)} \quad \text{.......(11)}$$

In the case of equal weights $wt_i = wt_j = 1/n$:

$$I = \frac{1}{n^2} \cdot \sum_{i=1}^{n} \left(2i - n\right) \cdot y^{(i)} \quad \text{.......(12)}$$

Where n is the number of original data. The weights could be calculated from any Declustering techniques.

Conclusion

Using generalized linear distribution (GLD) instead of uniform distribution in sequential Gaussian simulation has been presented. It allows us to simulate directly the P10, P50 and P90 or any interested ranked reservoir models based on the global distribution of the variable of interest. The idea is to calibrate the η parameter (slope parameter of GLD) which is only function of the *p*-value of interest and data. The analytical formula for η is derived. It only needs simple calculation for determining σ_m and I. We showed that η parameter has Gaussian distribution with mean of zero and standard deviation σ_m/I. The value of σ_m/I ensures that η to be in the range of -1 to 1. The final reservoir model has the global mean the same as the quantile value corresponding to the interested *p*-value from Gaussian distribution of global mean (Central Limit Theorem). This method honors the global uncertainty very well, increases the space of uncertainty (for each interested *p*-value we can create different number of realizations). This method shows that the traditional sequential Gaussian simulation just generate different realizations for the *p*-value of 0.5 (the *p*-value of 0.5 corresponds to the global mean of the Data because of Gaussianity).

The histogram and variogram reproduction are also checked. In the case of different p-values the range of variability of reproduced variogram and histogram is a function of distribution and range of variability of η parameter. In the case of variogram reproduction, high p-values will cause decreasing the variogram range in minor direction and increasing the variogram range in major direction.

Instead of using standard GLD other family of parametric distributions which gives values between 0 and 1 could be used too (*e.g.* beta distribution, etc). These families of parametric distribution have more than one parameter to be characterized and finding the relationship for them is not straightforward because of the complexity of mathematical formulas for these types of distributions. The usefulness of GLD is because of having one parameter (η) and simple closed form formulas for PDF, CDF and quantile function. The simplicity and linear behavior of standard GLD is useful in order to target the desired global mean with a certain percentile.

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Appendix: Generalized Linear Distribution

The Generalized Liner Distribution (GLD) is a three-parameter (a,b,η) family of continuous probability distributions. a and b are the minimum and maximum values (these two parameters can be any value) and η is the slope or shape parameter (which should be between -1 and 1). The distribution can be abbreviated as $L(a,b,\eta)$. The characterization of GLD is summarized as below:

Probability density function (*PDF*)*:*

The PDF of the generalized linear distribution is:

$$f(x;a,b,\eta)=\begin{cases}\dfrac{2\eta x+\left[(1-\eta)b-(1+\eta)a\right]}{(b-a)^2} & for \quad a\le x\le b\\ 0 \qquad otherwise & \end{cases}$$

Where

$$a,b\in(-\infty,+\infty)$$
$$\eta\in[-1,1]$$

Cumulative distribution function (*CDF*)*:*

The CDF of the generalized linear distribution is:

$$F(x;a,b,\eta)=\begin{cases}0 \qquad for \quad x<a & \\ \dfrac{\eta\left(x^2-a^2\right)+\left[(1-\eta)b-(1+\eta)a\right](x-a)}{(b-a)^2} & for \quad a\le x\le b\\ 1 \qquad for \quad x>b & \end{cases}$$

The CDF has a quadratic feature since its PDF is linear. This feature allows calculating the quantile function easily.

Quantile function:

From definition of quantile function we have:

$$q(p;a,b,\eta)=F^{-1}(p;a,b,\eta)$$

By using the CDF, the quantile function is calculated:

$$q(p;a,b,\eta)=\begin{cases}a+(b-a)p \qquad if \quad \eta=0 & \\ \left[\dfrac{(1+\eta)a-(1-\eta)b}{2\eta}\right]+\left(\dfrac{b-a}{2\eta}\right)\sqrt{(1-\eta)^2+4\eta p} & otherwise\end{cases}$$

Some of the important percentiles ($\eta\neq 0$):

- *Median* (*50*th *percentile*):

$$M=q(0.5;a,b,\eta)=\left[\frac{(1+\eta)a-(1-\eta)b}{2\eta}\right]+\left(\frac{b-a}{2\eta}\right)\sqrt{1+\eta^2}$$

- *Lower Quartile* (*25th percentile*):

$$q\left(0.25;a,b,\eta\right)=\left[\frac{\left(1+\eta\right)a-\left(1-\eta\right)b}{2\eta}\right]+\left(\frac{b-a}{2\eta}\right)\sqrt{1-\eta+\eta^2}$$

- *Upper Quartile* (*75^th^ percentile*):

$$q\left(0.75;a,b,\eta\right)=\left[\frac{\left(1+\eta\right)a-\left(1-\eta\right)b}{2\eta}\right]+\left(\frac{b-a}{2\eta}\right)\sqrt{1+\eta+\eta^2}$$

- *Interquartile Range:*

$$IR=q\left(0.75;a,b,\eta\right)-q\left(0.25;a,b,\eta\right)=\left(\frac{b-a}{2\eta}\right)\left[\sqrt{1+\eta+\eta^2}-\sqrt{1-\eta+\eta^2}\right]$$
$$=\frac{b-a}{\sqrt{1+\eta+\eta^2}+\sqrt{1-\eta+\eta^2}}$$

Mode:

$$Mode=\begin{cases} a & for \quad -1\le\eta<0 \\ \text{any value in } \left[a,b\right] & for \quad \eta=0 \\ b & for \quad 0<\eta\le 1 \end{cases}$$

Moments:

- *Mean* (*Expected value, 1^st^ non-centered moment*):

$$m=E\left\{X\right\}=\int_{-\infty}^{+\infty}x\cdot f\left(x;a,b,\eta\right)\cdot dx=\int_a^b x\cdot\frac{2\eta x+\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]}{\left(b-a\right)^2}\cdot dx$$
$$=\frac{2\eta}{\left(b-a\right)^2}\int_a^b x^2\cdot dx+\frac{\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]}{\left(b-a\right)^2}\int_a^b x\cdot dx$$
$$=\frac{\left(3+\eta\right)b+\left(3-\eta\right)a}{6}$$

- *2^nd^ non-centered moment*:

$$E\left\{X^2\right\}=\int_{-\infty}^{+\infty}x^2\cdot f\left(x;a,b,\eta\right)\cdot dx=\int_a^b x^2\cdot\frac{2\eta x+\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]}{\left(b-a\right)^2}\cdot dx$$
$$=\frac{2\eta}{\left(b-a\right)^2}\int_a^b x^3\cdot dx+\frac{\left[\left(1-\eta\right)b-\left(1+\eta\right)a\right]}{\left(b-a\right)^2}\int_a^b x^2\cdot dx$$
$$=\frac{\left(2+\eta\right)b^2+2ab+\left(2-\eta\right)a^2}{6}$$

- *Variance* (*2^nd^ centered moment*):

$$\mu_2 = E\left\{(X-m)^2\right\} = E\left\{X^2\right\} - m^2$$
$$= \frac{(\eta+2)b^2 + 2ab + (2-\eta)a^2}{6} - \left[\frac{(3+\eta)b + (3-\eta)a}{6}\right]^2$$
$$= \frac{(b-a)^2\left(3-\eta^2\right)}{36}$$

- *Non-centered moment of order k*:

$$E\left\{X^k\right\} = \int_{-\infty}^{+\infty} x^k \cdot f(x;a,b,\eta) \cdot dx = \int_a^b x^k \cdot \frac{2\eta x + \left[(1-\eta)b - (1+\eta)a\right]}{(b-a)^2} \cdot dx$$
$$= \frac{2\eta}{(b-a)^2}\int_a^b x^{k+1} \cdot dx + \frac{\left[(1-\eta)b-(1+\eta)a\right]}{(b-a)^2}\int_a^b x^k \cdot dx$$
$$= \frac{\left[(1+\eta)k+2\right]b^{k+2} - (k+2)(1+\eta)ab^{k+1} - (k+2)(1-\eta)a^{k+1}b + \left[(1-\eta)k+2\right]a^{k+2}}{(k+1)(k+2)(b-a)^2}$$

- *Centered moment of order k*:

$$\mu_k = E\left\{(X-m)^k\right\} = \int_{-\infty}^{+\infty} (x-m)^k \cdot f(x;a,b,\eta) \cdot dx$$
$$= \int_{-\infty}^{+\infty}\left[\sum_{i=0}^{k} C_k^i \cdot (-m)^{k-i} \cdot x^i\right] \cdot f(x;a,b,\eta) \cdot dx = \sum_{i=0}^{k} C_k^i \cdot (-m)^{k-i} \cdot \underbrace{\int_{-\infty}^{+\infty} x^i \cdot f(x;a,b,\eta) \cdot dx}_{=E\left\{X^i\right\}}$$
$$= \sum_{i=0}^{k} C_k^i \cdot (-m)^{k-i} \cdot E\left\{X^i\right\}$$

We know that

$$E\left\{X^i\right\} = \frac{\left[(1+\eta)i+2\right]b^{i+2} - (i+2)(1+\eta)ab^{i+1} - (i+2)(1-\eta)a^{i+1}b + \left[(1-\eta)i+2\right]a^{i+2}}{(i+1)(i+2)(b-a)^2}$$
$$m = E\{X\} = \frac{(3+\eta)b + (3-\eta)a}{6}$$

Therefore

$$\mu_k = \sum_{i=0}^{k}\left[(-1)^{k-i} \cdot C_k^i \cdot \left(\frac{b(3+\eta)+a(3-\eta)}{6}\right)^{k-i} \cdot \frac{\left[(1+\eta)i+2\right]b^{i+2} - (i+2)(1+\eta)ab^{i+1} - (i+2)(1-\eta)a^{i+1}b + \left[(1-\eta)i+2\right]a^{i+2}}{(i+1)(i+2)(b-a)^2}\right]$$

- *Skewness:*

$$\gamma_1 = \frac{\mu_3}{\mu_2^{1.5}}$$

- *Excess kurtosis:*

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3$$

Moment Generating Function:

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} \cdot f(x;a,b,\eta) \cdot dx = \int_a^b e^{tx} \cdot \frac{2\eta x + \left[(1-\eta)b - (1+\eta)a\right]}{(b-a)^2} \cdot dx$$

$$= \frac{2\eta}{(b-a)^2} \int_a^b x e^{tx} dx + \frac{\left[(1-\eta)b - (1+\eta)a\right]}{(b-a)^2} \int_a^b e^{tx} dx$$

$$= \left(\frac{2\eta}{(b-a)^2}\right)\left(\frac{be^{bt} - ae^{at}}{t}\right) - \left(\frac{2\eta}{(b-a)^2} - \frac{\left[(1-\eta)b - (1+\eta)a\right]}{(b-a)^2} t\right)\left(\frac{e^{bt} - e^{at}}{t^2}\right)$$

Information Entropy:

$$H\{X\} = \int_{-\infty}^{+\infty} f(x) \cdot \ln\left[f(x)\right] \cdot dx = \int_a^b f(x) \cdot \ln\left[f(x)\right] \cdot dx$$

$$H\{X\} = \begin{cases} \frac{-1}{(b-a)^2}\left[2\ln\left(\frac{2}{b-a}\right) - 1\right] & \textit{for} \quad \eta = -1 \\ \ln(b-a) & \textit{for} \quad \eta = 0 \\ \frac{1}{(b-a)^2}\left[2\ln\left(\frac{2}{b-a}\right) - 1\right] & \textit{for} \quad \eta = 1 \\ \frac{1}{2}\left[\left(\frac{1+\eta}{b-a}\right)^2 \ln\left(\frac{1+\eta}{b-a}\right) - \left(\frac{1-\eta}{b-a}\right)^2 \ln\left(\frac{1-\eta}{b-a}\right)\right] - \frac{\eta}{(b-a)^2} & \textit{otherwise} \end{cases}$$

Special cases:

- *Uniform distribution between a, b; U(a,b)*

  For *U*(*a,b*) the parameter η is equal to zero (zero slope):

$$f(x;a,b,0) = \begin{cases} \frac{1}{(b-a)} & \textit{for} \quad a \le x \le b \\ 0 & \textit{otherwise} \end{cases}$$

- *Standard generalized linear distribution:*

  In this case, the parameters *a* and *b* are equal to zero and one respectively, therefore the PDF, CDF and quantile function can be summarized as below:

$$f\left(x;0,1,\eta\right)=\begin{cases}2\eta x+\left(1-\eta\right) & for \quad 0\le x\le 1\\ 0 & otherwise\end{cases}$$

$$F\left(x;0,1,\eta\right)=\begin{cases}0 & for \quad x<a\\ \eta x^2+\left(1-\eta\right)x & for \quad a\le x\le b\\ 1 & for \quad x>b\end{cases}$$

$$q\left(p;0,1,\eta\right)=\frac{\sqrt{\left(1-\eta\right)^2+4\eta p}+\eta-1}{2\eta}$$