Application of Location-Dependent Moments in Estimation and Simulation

David F. Machuca-Mory and Clayton V. Deutsch

The use of location-dependent moments and distributions in estimation and simulation is proposed under an assumption of local stationarity. Location dependent distributions and variograms are informed by exhaustive maps of their local model parameters. They can be used for locally stationary simple and ordinary kriging as well as for change of support models with local spatial continuity. Multigaussian kriging with local prior distributions is achieved by modelling the local normal scores transformation function using local Hermite polynomials. When samples are clustered or moderate in number, estimation with location dependent variograms shows an increased accuracy, precision and reproduction of local spatial features. Estimation with local prior distributions improves greatly the precision but at the price of decreasing the accuracy. If abundant and evenly spaced data is available, the information provided by these samples outweighs the importance of local measures of continuity and prior distributions. Significant computer resources are required to vary all parameters.

Introduction

Using locally varying moments and distributions for the spatial modelling of a geologic or environmental attribute allows introducing a locally relevant, rather than globally relevant, background information basis. The procedure for calculating locally varying parameters was presented earlier in this report (Machuca-Mory & Deutsch, 2008a,b). In classic geostatistics, the range of possible values that an attribute can take at a unsampled location **u** is modelled by the distribution of a stationary Random Variable $Z(\mathbf{u})$.

$$\begin{gathered}\Pr\left\{Z(\mathbf{u}_i)<z_1,...,Z(\mathbf{u}_n)<z_K\right\}=\Pr\left\{Z(\mathbf{u}_i+\mathbf{h})<z_1,...,Z(\mathbf{u}_n+\mathbf{h})<z_K\right\}\\ \forall\ \mathbf{u}_i,\mathbf{u}_i+\mathbf{h}\in\mathcal{D}\end{gathered}\tag{1}$$

In contrast, if local stationarity is assumed, the moments and the multivariate cdf of the random function is invariant by translation only if they are defined in relation to the same reference or anchor point **o** Thus,

$$\begin{gathered}\Pr\left\{Z(\mathbf{u}_i)<z_1,...,Z(\mathbf{u}_n)<z_K;\mathbf{o}_\alpha\right\}=\Pr\left\{Z(\mathbf{u}_i+\mathbf{h})<z_1,...,Z(\mathbf{u}_n+\mathbf{h})<z_K;\mathbf{o}_\beta\right\}\\ \forall\ \mathbf{u}_i,\mathbf{u}_i+\mathbf{h}\in\mathcal{D},\text{ and only if }\ \alpha=\beta\end{gathered}\tag{2}$$

This location dependency is important for the assumption of local or quasi stationarity, and it is the major difference in relation to a fully non-stationary model:

$$\begin{gathered}\Pr\left\{Z(\mathbf{u}_i)<z_1,...,Z(\mathbf{u}_n)<z_n\right\}\neq\Pr\left\{Z(\mathbf{u}_i+\mathbf{h})<z_1,...,Z(\mathbf{u}_n+\mathbf{h})<z_n\right\}\\ \forall\ \mathbf{u}_i,\mathbf{u}_i+\mathbf{h}\in\mathcal{D}\end{gathered}\tag{3}$$

This paper explains the application of location-dependent moments and distributions in estimation and simulation under the assumption of local stationarity. Location-dependent variograms are applied to a locally stationary ordinary kriging and complement the information provided by the local mean in locally ordinary simple kriging. The increased local information basis provided by a locally changing histogram shape is exploited by a locally stationary multigaussian kriging and locally stationary sequential Gaussian simulation. The location-dependent variograms yield to estimation variances that are not only dependent of the local data arrangement, but also of the local spatial continuity. These enhanced estimation variances is used for P-field simulation under a globally stationary and multigaussian assumption. These estimation and simulation techniques are compared with techniques based in the traditional stationarity models.

Estimation with location-dependent moments

A locally varying mean has been incorporated in several types of kriging. Simple kriging with varying means (Deutsch & Journel, 1998) uses a prior model of the local means at every location. The simple kriging (SK) estimator and system of equations are reproduced here because they are akin to those of kriging with location-dependent moments. Thus, the SK estimator is expressed as:

$$Z^*_{SK}(\mathbf{u}) - m(\mathbf{u}) = \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{u})\left[Z(\mathbf{u}_\alpha) - m(\mathbf{u}_\alpha)\right] \tag{4}$$

Where the weights λ_α are obtained by simple kriging (Deutsch & Journel, 1998; Goovaerts, 1997):

$$\sum_{\beta=1}^{n} \lambda_\beta(\mathbf{u}) C(\mathbf{u}_\beta, \mathbf{u}_\alpha) = C(\mathbf{u}, \mathbf{u}_\alpha) \quad \alpha = 1,\ldots,n \tag{5}$$

Where $C(\mathbf{u}_\beta, \mathbf{u}_\alpha)$ and $C(\mathbf{u}, \mathbf{u}_\alpha)$ are the covariances between sample pairs, and between samples and the estimated point **u**. Techniques such as universal kriging (Goovaerts, 1997) incorporate a deterministic trend model of the form :

$$m(\mathbf{u}) = \sum_{k=0}^{K} a_k f_k(\mathbf{u}) \tag{6}$$

Where $f_k(\mathbf{u})$ are a set of known functions (usually polynomials) of the coordinates, and a_k are unknown coefficients that are implicitly estimated in the universal kriging system of equations (Deutsch & Journel, 1998). In kriging with location-dependent moments, not only the mean varies in function of the location, but also the spatial correlation measures. Additionally, by using a local normal scores transformation it is possible to honour the local histogram shape in the back transformation of estimates. These proposed techniques are described next.

Simple and ordinary kriging with LDM

Under a locally stationary decision the SK estimator becomes:

$$Z^*_{LSSK}(\mathbf{o}) = \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{o}) Z(\mathbf{u}_\alpha) + \left[1 - \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{o})\right] m(\mathbf{o}) \tag{7}$$

Which is similar to the stationary form of SK, but with the difference that the mean is specific of the estimated point **o**. Note that the notation of the estimated point location is changed from **u** to **o**, this is done in order to stress that **o** not only the estimated location but it is also the point in relation to which all the location-dependent moments are defined. The locally stationary simple kriging (LSSK) allows the use of location-dependent covariances (or correlograms):

$$\sum_{\beta=1}^{n} \lambda_\beta(\mathbf{o}) C(\mathbf{u}_\beta - \mathbf{u}_\alpha; \mathbf{o}) = C(\mathbf{o} - \mathbf{u}_\alpha; \mathbf{o}) \quad \alpha = 1,\ldots,n \tag{8}$$

The LSSK variance is given by:

$$\sigma^2_{LSSK}(\mathbf{o}) = C(0;\mathbf{o})\left[1 - \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{o}) \rho(\mathbf{o} - \mathbf{u}_\alpha; \mathbf{o})\right] \tag{9}$$

Where $C(0;\mathbf{o})$ is the location-dependent covariance at lag 0, which can be demonstrated that is equivalent to location-dependent variance $\sigma^2(\mathbf{o})$ under a consistent set of weights for location-dependent 1-point and 2-point moments (Machuca-Mory & Deutsch, 2008a; 2008b). Thus, the LSSK variance is locally conditioned not only by the data availability and arrangement around the estimated point, but also by the local variability, which informed by the location-dependent variance, as well as by the local spatial correlation, which is informed by the location-dependent covariance or correlogram.

The locally stationary ordinary kriging (LSOK) estimator is similar to the traditional ordinary kriging estimator (Deutsch & Journel, 1998):

$$Z^*_{LSOK}(\mathbf{o}) = \sum_{\alpha=1}^{n} \lambda_\alpha^{(OK)}(\mathbf{o}) Z(\mathbf{u}_\alpha) \tag{10}$$

But the covariances in the LSOK system of equations are location-dependent:

$$\begin{cases} \sum_{\beta=1}^{n} \lambda_\beta^{(OK)}(\mathbf{o}) C(\mathbf{u}_\beta - \mathbf{u}_\alpha; \mathbf{o}) + \mu(\mathbf{o}) = C(\mathbf{o} - \mathbf{u}_\alpha; \mathbf{o}) \quad \alpha = 1,\ldots,n \\ \sum_{\beta=1}^{n} \lambda_\beta^{(OK)}(\mathbf{o}) = 1 \end{cases} \tag{11}$$

As in the case of LSSK, the LSOK variance is enhanced by the incorporation of the location-dependent variance and covariances:

$$\sigma^2_{LSOK}(\mathbf{o}) = C(0;\mathbf{o}) - \sum_{\alpha=1}^{n} \lambda_\alpha^{(OK)}(\mathbf{o}) C(\mathbf{o} - \mathbf{u}_\alpha; \mathbf{o}) - \mu(\mathbf{o}) \tag{12}$$

These locally stationary kriging forms allow the incorporation of location-dependent means, variances and 2-point measures of correlation in the estimation process.

Locally stationary multigaussian kriging

Under the multigaussian approach the inference of the posterior local cdf reduces only to the estimation of the local mean and variance (Goovaerts, 1997). The non-gaussianity of the univariate distribution of most attributes is not a major issue, since this can be normal-scores transformed (Deutsch & Journel, 1998). However, the bivariate gaussianity cannot be assured by the normal scores transformation of the global distribution and multivariate gaussianity must to be assumed in most cases (Verly, 1983). Beyond the multigaussian assumption, this approach requires a decision of strict stationarity, thus everywhere within a domain the prior univariate and multivariate distributions are assumed normal (0,1). Moreover, since the normal scores transformation is performed over the global original values distribution, the desicion of strict stationarity of the multigaussian distribution implicitly translates in an assumption of strict stationarity of the original values distribution. This stringent assumption may be deemed inadequate in presence of strong non-stationary features that not only affect the mean but also the variance and cdf shape. Additionally, the posterior local Gaussian distributions is obtained from the SK estimate and variance, which means that the local data variability is not incorporated in the posterior distributions in original units after back-transformation.

In the locally stationary multigaussian approach, the prior local distributions are obtained by weighting the available samples inversely to their distances to a reference point **o** (Machuca-Mory & Deutsch, 2008a), subsequently, these local distributions are locally normal scores transformed. The resultant location-dependent local scores transformation functions are expressed as:

$$Z(\mathbf{u}_i) = F^{-1}\left(G(Y(\mathbf{u}_i);\mathbf{o}\right) = \varphi_Z(Y(\mathbf{u}_i);\mathbf{o}) \qquad i = 1,\ldots,n \tag{13}$$

In order to reduce dimensionality the local normal scores transformation functions are not described by correspondence tables between original and transformed values (transformation tables), but modelled by Hermitian polynomial series of at most 30 coefficients and terms.

The back-transformation of this distribution is performed using the reconstructed transformation tables. This back-transformation honours the location-dependent distribution. Thus, the variance of the posterior distribution in original units is affected not only by the stationary covariance and the samples arrangement around the estimated points, but also by the local variance, local cdf shape and local spatial continuity.

So far the locally stationary simple, ordinary and multigaussian kriging techniques described are related to point estimation. The first two can be easily extended to locally stationary block estimation, while the last one needs a change of support model for the estimated values.

Software implementation for estimation with location-dependent moments

The FORTRAN program `KT3D_MGlp` allows LSSK and LSOK with location-dependent variograms and multigaussian. The parameter file for this program is an extension of the `KT3D` parameter file (see figure 2). A line for the file containing the interpolated Hermite coefficients for local Gaussian transformation has been added. The number and starting column for these coefficients is specified in the next line. If a valid file is not provided, data transformation will not be performed. This is adequate for estimation with a global cdf in original or normal units. The last files below the specifications of the globally stationary variogram model correspond to the gridded parameters of the location-dependent variograms with up to two structures. If one or more of these parameters are not provided their value is replaced by the corresponding of the globally stationary variogram definition.

Simulation with location-dependent moments

The use of location-dependent moments can be extended to simulation. As in the case of estimation, the main difference compared with stationary techniques is the introduction of a location-dependent moments under the locally stationarity decision. This is translated in a considerable increase of the computational demand for the application of the simulation algorithms with locally changing parameters, although the core of the methodology remains similar as in the stationary case.

Sequential Gaussian simulation with LDM

Given a locally normal scores transformed values and a random path visiting each simulation node, $\mathbf{o}'$, only once, the core of the sequential Gaussian (SGS) algorithm remains unchanged, this is (Goovaerts, 1997):

- At a node $\mathbf{o}'$ in the random path define the Gaussian conditional cdf from the mean and variance resulting of applying SK using normal scores data and previously simulated values within a search neighbourhood.
- Generate a random value $y^{(l)}(\mathbf{o}')$ from such distribution and add to the dataset.
- Go to the next node in the random path, repeat previous steps and loop until all nodes are simulated.
- Back-transform simulated values
- Repeat all steps with a different random path for generating another realization.

The challenge of using location-dependent moments and distributions is that at each simulation node, $\mathbf{o}'$, the variogram model used in LSSK may change, so does the back-transformation function. This greatly increases the computer memory requirements since the location-dependent variogram model parameters and Hermite coefficients of the local back-transformation function must be stored for each simulation node. Moreover other artifices designed to reduce the processing demand of the sequential simulation algorithm cannot be applied. An example of these artifices is the covariance lookup table, which stores the data-data and data-point covariance matrices for all the nodes located within twice the search radius (Deutsch & Journel, 1998). Under the strict stationary decision, the covariance lookup table needs to be calculated only once, which means a considerable saving in processing time. By contrast, when using location-dependent parameters, these covariances need to be calculated at each node $\mathbf{o}'$.

The huge memory requirements of locally stationary sequential Gaussian simulation (LSSGS) can be mitigated by following a sequential path instead of a random path. So, the location-dependent parameters can be read sequentially from external files. However, this may cause artifacts to appear in the simulated maps (Deutsch, 2002). Alternatively, the location-dependent moments can be stored only for the anchor points where these were calculates, and obtained by interpolation at each node during the simulation. This would reduce the memory storage requirements, but increase the processing time.

Another option for reducing the excessive memory and computer processing demand is to define the location parameters not at the resolution of simulation, which is concordant to the sample scale, but at the block scale.

P-field simulation with LDM

The p-field simulation greatly reduces the processing and memory requirements of simulation with location-dependent moments. In fact, this simulation technique takes the same resources as in the stationary case. This is because the local conditional distributions are modelled only once by multigaussian LSSK, rather than for every realization. These conditional distributions are constructed only with original data, and instead of random Monte Carlo drawing, simulated values are obtained from correlated probabilities (Deutsch, 2002). Thus, the computer memory and processing requirements are considerably decreased.

This very efficient simulation methodology has however some important drawbacks that have caused p-field simulation be dismissed as an appropriate simulation method for continuous variables (Deutsch, 2002). These challenges are caused by the inherent characteristics of the methodology and appear as two kinds of artifacts in the proximity of hard data locations (Pyrcz & Deutsch, 2001). The first artifact appears as tendency of hard data values to appear as a local extreme. The second artifact is the increased continuity in the proximity of conditioning hard data, which introduces a covariance bias hindering the variogram reproduction of simulated values.

Despite these drawbacks in p-field simulation and the warnings about its use, this remains the only viable simulation method with location-dependent moments and distributions for large datasets on 32-bit computers. Currently the correlation between simulated values is induced by a globally stationary covariance. A p-field simulation algorithm that incorporates locally stationary covariances needs to be developed.

Example

Again, the samples used for illustrating the application of Multigaussian Kriging with location-dependent moments were taken from exhaustive Walker Lake data set (Isaaks & Srivastava, 1989). This data set and the clustered and gridded subsets are shown in Figure 3.

Also, the same location-dependent correlogram model parameters and the exhaustive Hermite coefficients for local transformation that were presented earlier are used for this example. Four varieties of Multigaussian Kriging are compared: 1) Multigaussian simple kriging with globally stationary cdf and variogram (MGSK), 2) Multigaussian simple kriging with globally stationary cdf but location-dependent variograms (MGLSSK with LDV), 3) Multigaussian simple kriging with location-dependent cdf 's but globally stationary variogram (MGLSSK), 4) Multigaussian simple kriging with location-dependent cdf 's and variograms (full-MGLSSK).

These techniques were applied for both the clustered and gridded data sets. In the case of the gridded data the multigaussian kriging with location-dependent variograms was tested with a fixed exponential variogram shape, and with stable variograms with locally varying shape. The stationary model of $1-\rho(h)$ for the clustered data set is presented in table 1, while the corresponding exponential and stable models for the gridded data set is presented in table 3.

The different types of multigaussian kriging were performed at pixel scale and without block discretization. This means only point estimation was done for this example. Figure 4 present the post-processed posterior local means of 4 types of multigaussian kriging performed on the clustered data set. There, it can be observed the increased difference between high grade and low grade areas introduced by the use of local cdf's. It can be observed as well a better reproduction of the local spatial features when location-dependent correlograms are used. These locally varying patterns of spatial continuity produce a map that is more concordant with the exhaustive data set in figure 3. The variances in normal score units, before post-processing, are shown in figure 5. Note there that the variances of multigaussian kriging with a globally stationary correlogram model are dependent only of surrounding data configuration. Whereas, the variances obtained using location-dependent correlogram models in the estimation are affected by the local continuity informed by these models. After post-processing the posterior variances are modified by the local data variability. The influence of the local correlograms models is visible in the 2 of the posterior variance maps (right side of the figure 6), while the use of prior local cdf's translate in an increased contrast of the posterior local variances (bottom side of the figure 6).

For the clustered data set, crossvalidation and estimation results show that when locally stationary correlogram models are used the accuracy and precision of the estimates increase slightly compared with the classical multigaussian kriging results. This is shown by a reduction of 1.3% in the average square error and increase of 1.5% in the coefficient of correlation between true and estimated values when local variograms are used. While the smoothing of estimates is reduced, this is reflected by an increase of 7.9% in the variance of estimates.

When local prior cdf's are used, the local accuracy may suffer a slightly to moderate decrease, but local precision is increased. This is seen by the increase of the average square error (Table 2-a) and a decrease of the coefficient of correlation between true and estimated values (Table 2-b), whereas the smoothing of estimates is greatly reduced (Table 2-c), as well as the estimation variance average (Table 2-d).

Accuracy plots (Deutsch, 1996) for this data set were generated using normal estimates of the posterior mean and variance before post-processing (see Figure 7). Stationary multigaussian kriging show wide probabilitydistributions, these are increased when location-dependent correlograms are used. However the distributions are narrowed when local prior cdf's are introduced. The two effects are balanced in full multigaussian kriging with location-dependent variograms and cdf's, yielding slightly narrow distributions and a fairer reproduction of the original proportions in the posterior cdf's.

As mentioned before, the estimated models of the gridded data set were constructed using exponential and stable correlogram models, both globally and locally stationary. The globally stationary models using exponential and stable single structures are presented in Table 3.

Only the maps of post-processed estimated means using an exponential model are presented here (see Figure 8), since the maps produced using stable models are very similar. They do not show a clear improvement of local spatial features reproduction when locally stationary correlograms are used. This is because the abundant data available outweighs the information input supplied by the location-dependent correlograms. The semi-regular and dense sampling pattern yields to low estimation variances everywhere when a globally stationary correlogram model is used (see Figure 9, left column). The regular pattern of the spatial distribution of these variances reflects the samples semi-regular pattern. By contrast, when a location-dependent correlograms are used the estimation variances in normal units reflect the local changes of spatial continuity (see Figure 9, right column). This is more patent when a stable location-dependent model is used, as it can be observed in the right column of Figure 10. There, the highly continuous areas of low estimation variances correspond to areas where the local stable model has a higher power value, and vice versa.

The statistics of cross validation and estimation results for gridded data (see Tables 3 and 5) show the same features as those obtained from the clustered data set. This is a decrease of local accuracy outweighed by an increase of local precision when location-dependent correlograms and prior cdf's are used. This effect is clearer when stable location-dependent correlograms models are used (see Table 5).

The abundance of semi-regularly spaced data causes a considerable narrowing of the posterior distributions when a global cdf is considered (see top line in Figure 11). Contrarily to the case of the clustered data set, the introduction of location-dependent cdf's makes the posterior distributions wider, correcting in this way the reproduction of original proportions (see bottom line in Figure 11).

Discussion and Conclusions

Estimation and simulation with location-dependent moments and distributions is possible under a decision of local stationarity. These techniques require the prior exhaustive models of the local moments, and in the case of multigaussian kriging, of the Hermitian coefficients that approximate the local normal scores transformation functions. This greatly increases the work of parameter inference as well as computer requirements for estimation and simulation. I n particular, the memory and processing requirements for sequential Gaussian conditional simulation with random paths can become prohibitive.

Multigaussian kriging with location-dependent moments and distributions is feasible without an excessive demand of computer resources. Its advantage is greater when applied on clustered or sparse data, particularly in areas were local data scarcity may be complemented by the information provided by local measures of continuity and prior cdfs. If data is abundant and evenly spaced, the increased pre-processing and estimation requierements of using location-dependent moments and distributions may be not worth and stationary methods may be preferred. Similarly, if samples are very scarce and there is no other information source for obtaining the local moments, there is no other option than adopt the classic stationarity assumptions.

According to the cross-validation results and the statistics of estimates, location-dependent variograms can increase the accuracy and precision of estimates on clustered data. The incorporation of prior local cdfs may decrease the local accuracy in favour of a high local precision.

When a dense quasi-regular grid of data is used, the use of location-dependent correlograms may decrease both the local accuracy and precision. Using local prior cdfs improves greatly the local precision in this case. This decrease in local accuracy can be explained by the prevailing importance of the information provided by the dense sampling over the information provided by local variograms. However, a dense sampling pattern can yield to a misleadingly narrow and uniform posterior distributions estimated with a globally stationary correlogram. The use of locally stationary correlograms and cdfs may correct this by allowing areas of differentiated spatial variability.

Some issues of multigaussian kriging with location-dependent moments and distributions still need further research. One of them is the abundance of posterior local distributions with very low variance, particularly when local stable models are used. Another is the algorithm for local change of support in order to obtain the distributions of the block scale.

Beyond the virtues and challenges of estimation and simulation with location-dependent moments and distributions discussed here, these techniques show and interesting promise for enhancing the geological realism of numerical models. These improved models should translate in improved decisions taken by using them.

References

Chilès, J.-P., & Delfiner, P. (1999). *Geostatistics, modeling spatial uncertainty.* New York: John Wiley & Sons, INC.

Deutsch, C. V. (1996). Directc assessment of local accuracy and precision. In E. Y. Baafi, & N. A. Schofield (Ed.), *Geostatistics Wollongong '96. 1*, pp. 115-125. Wollongong: Kluwer Academic.

Deutsch, C. V. (2002). *Geostatistical Reservoir Modeling.* New York: Oxford University Press.

Deutsch, C. V., & Journel, A. G. (1998). *GSLIB. Geostatistical software library and user's guide* (2nd Edition ed.). New York: Oxford University Press.

Goovaerts, P. (1997). *Geostatistics for Natural Resources Evaluation.* New York: Oxford University Press.

Isaaks, E. H., & Srivastava, R. M. (1989). *An introduction to applied geostatistics.* New York: Oxford University Press.

Machuca-Mory, D. F., & Deutsch, C. V. (2008b). *Location-dependent moments and distributions based in continously varyng weights - Paper 109.* Centre for Computational Geostatistics - Report 10.

Machuca-Mory, D. F., & Deutsch, C. V. (2008a). *Optimal weights for location-dependent moments - Paper 108.* Centre for Computational Geostatistics - Report 10.

Matheron, G. (1970). *La Théorie des variables régionalisées et ses applications.* . Les cahiers du CMM, Fasc. 5, Ecole des Mines de Paris, Fontainebleau.

Pyrcz, M. J., & Deutsch, C. V. (2001). Two artifacts of probability field simulation. *Mathematical Geology , 33* (7), 775-799.

Sans, H., & Martin, V. (1984). Technical parametrisation of uranium reserves to be mined by open-pit method. In G. Verly, M. David, A. G. Journel, & A. Marechal (Ed.), *Geostatistics for Natural Resources Characterization. 2*, pp. 1071-1085. Dordrecht: Reidel Publishing Company.

Vann, J., & Sans, H. (1995). Global resource estimation and change of support at the Enterprise Gold Mine, Pine Creek, Northern Territory - Application of the geostatistical discrete gaussian model. *Applications of Computers and Operations Research in the Mineral Industry APCOM XXV*, (pp. 171-179). Brisbane.

Verly, G. (1983). The multigaussian approach and its applications to the estimation of local reserves. *Mathematical Geology , 15* (2), 259-286.

Table 1: Stationary exponential 1-ρ(h) model parameters for clustered data

Nugget effect	sill contribution	Anisotropy rotation	Maximum range	Minimum range
0.25	0.75	165	119	28

Table 2: Summary of cross-validation and estimation results for the clustered data set

CDF	Correlogram	MSE	ρ (est-true)	Var of Est	Avg K-Var
Global	Global	36980	0.77	33190	39515
Global	Local	36515	0.78	33200	39665
Local	Global	42250	0.73	40250	26400
Local	Local	43620	0.74	42000	26000

Table 3: Stationary exponential and stable 1-ρ(h) models parameters for gridded data

Variogram model	Nugget effect	Model exponent	sill contribution	Anisotropy rotation	Maximum range	Minimum range
Exponential	0.00	1.00	1.00	160	90	37
Stable	0.00	1.00	1.28	160	88	34

Table 4: Summary of cross-validation and estimation results for the gridded data set

CDF	Correlogram	MSE	ρ (est-true)	Var of Est	Avg K-Var
Global	Global	22300	0.81	50300	16350
Global	Local	22680	0.81	51350	21400
Local	Global	25850	0.77	50800	11900
Local	Local	25250	0.78	51650	14750

Table 5: Summary of cross-validation and estimation results with stable variogram

CDF	Correlogram	MSE	ρ (est-true)	Var of Est	Avg K-Var
Global	Global	22950	0.80	52300	12750
Global	Local	23750	0.79	54850	14150
Local	Global	26750	0.76	53200	9300
Local	Local	26950	0.76	55650	8800

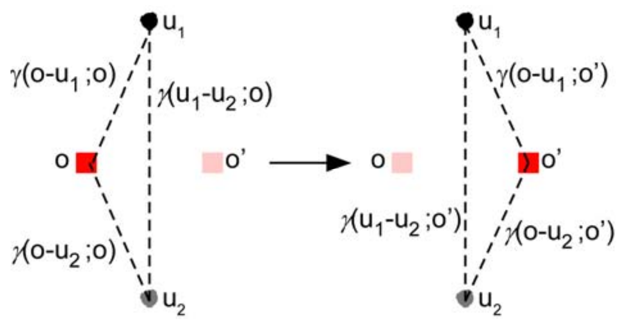

Figure 1: Estimation under the assumption of local stationarity and with location-dependent variograms

```
START OF PARAMETERS:
samples.out                            - file with data
0  1  2  0  8  0                       - columns for DH,X,Y,Z,var,sec var
-10.0   1.0e21                         - trimming limits
Herpol-loc.dat                         - file with local Hermite polynomials (in columns)
30 1                                   - number of hermite polynomials and column for phi(0)
0   1530                               - minimum and maximum value in original units
0                                      - option: 0=grid, 1=cross, 2=jackknife
xvk.dat                                - file with jackknife data
1   2   0    3    0                    - columns for X,Y,Z,vr and sec var
1                                      - debugging level: 0,1,2,3
MGkt3d-lssk-exp.dbg                    - file for debugging output
MGkt3d-lssk-exp.out                    - file for kriged output
260   0.5   1                          - nx,xmn,xsiz
300   0.5   1                          - ny,ymn,ysiz
1    0.0    1.0                        - nz,zmn,zsiz
1    1      1                          - x,y and z block discretization
4    34                                - min, max data for kriging
0                                      - max per octant (0-> not used)
150.0   150.0  1.0                     - maximum search radii
 0.0    0.0   0.0                      - angles for search ellipsoid
0       0.000                          - 0=SK,1=OK,2=non-st SK,3=exdrift, 4 =Locally st. SK
0 0 0 0 0 0 0 0 0                      - drift: x,y,z,xx,yy,zz,xy,xz,zy
0                                      - 0, variable; 1, estimate tren
extdrift.dat                           - gridded file with drift/mean
0                                      - column number in gridded file
1    0.25   1.0                        - nst, nugget effect
2    0.75 165.0    0.0    0.0          - it,cc,ang1,ang2,ang3
          119.0  28.0  10.0            - a_hmax, a_hmin, a_vert
1sexp-c0.out                           - local nugget effect (same grid)
1sexp-exp.out                          - local stable model power (same grid)
1sexp-c1.out                           - local sill of 1st structure (same grid)
1sexp-ahmax1.out                       - local range hmax of 1st structure (same grid)
1sexp-ahmin1.out                       - local range hmin of 1st structure (same grid)
1sexp-ahver.out                        - local range vertical of 1st structure (same grid)
1sexp-ang1.out                         - local angle 1 of 1st structure (same grid)
1sexp-ang2.out                         - local angle 2 of 1st structure (same grid)
1sexp-ang3.out                         - local angle 3 of 1st structure (same grid)
2sexp-c2.out                           - Local sill for 2nd structure
2sexp-hmax.out                         - Local maximum range for 2nd structure
2sexp-hmin.out                         - Local minimum range for 2nd structure
2sexp-hmin.out                         - Local vertical range for 2nd structure
2sexp-ang1.out                         - Local angle 1 for 2nd structure
2sexp-ang2.out                         - Local angle 2 for 2nd structure
2sexp-ang3.out                         - Local angle 3 for 2nd structure
```

Figure 2: Example of the parameter file for `KT3D_MGlp`

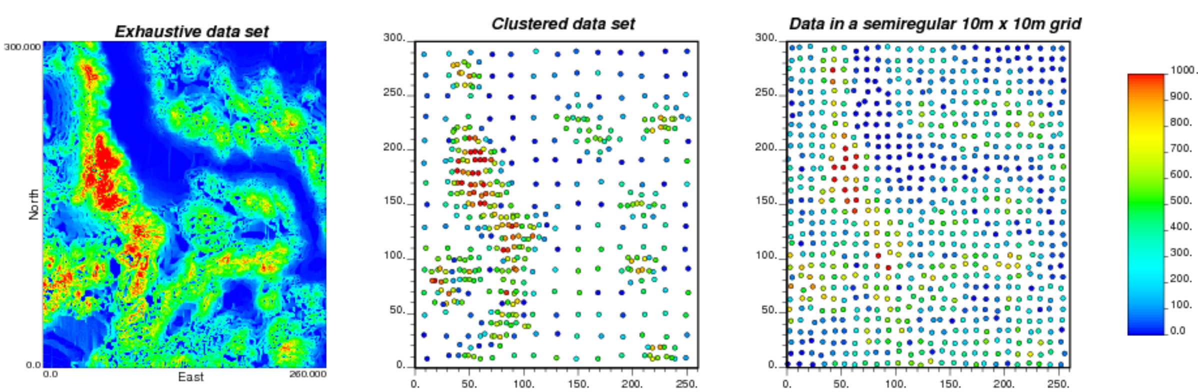

Figure 3: Exhaustive, clustered and gridded datasets of the Walker Lake site.

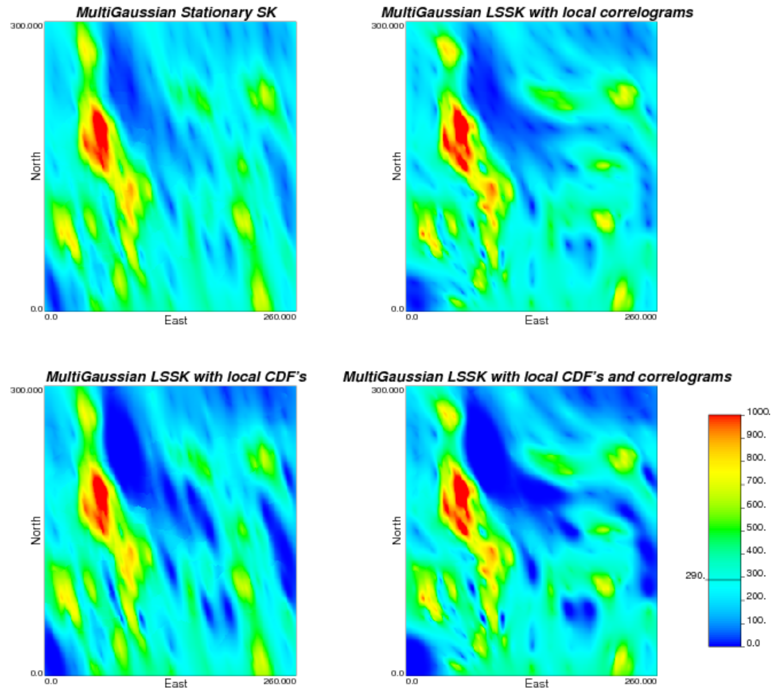

Figure 4: Postprocessed local means obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the clustered dataset.

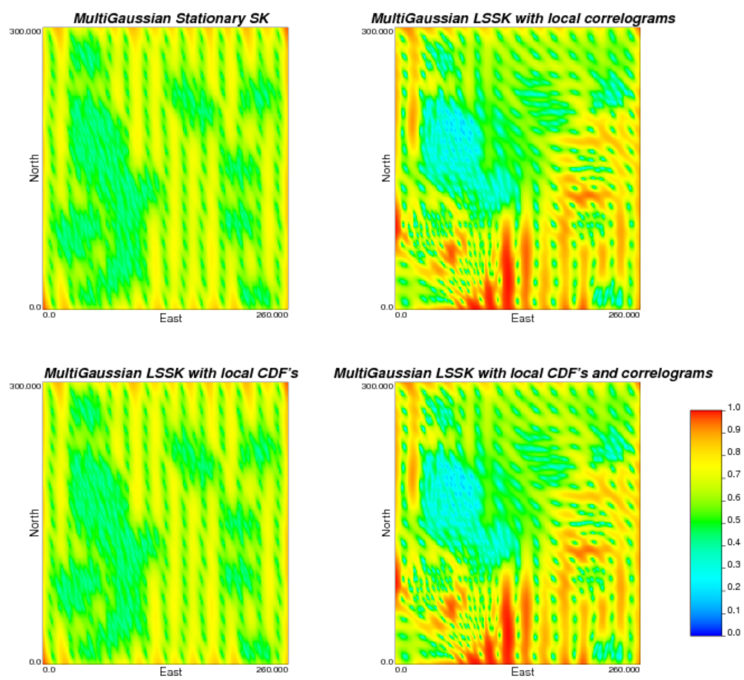

Figure 5: Estimation variances in normal units obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the clustered dataset.

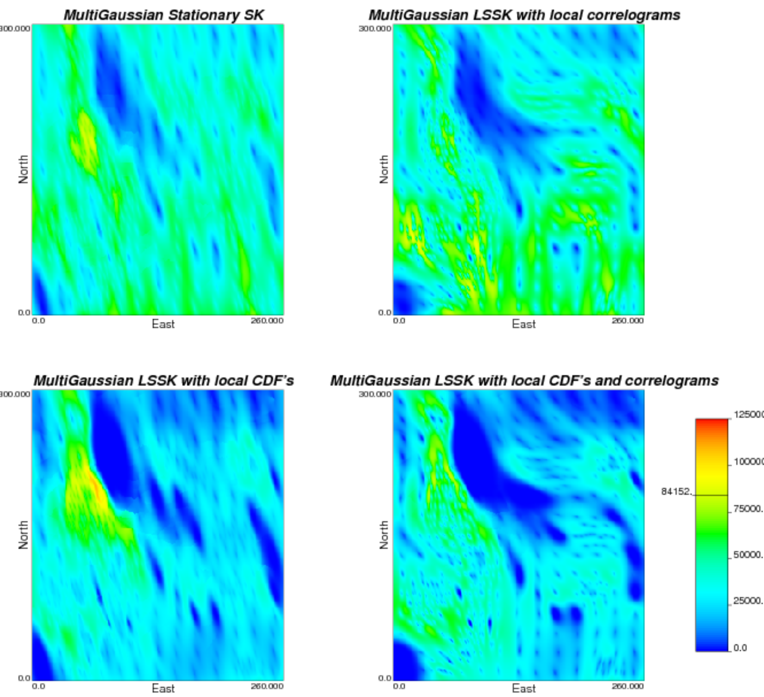

Figure 6: Postprocessed local variances obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the clustered dataset.

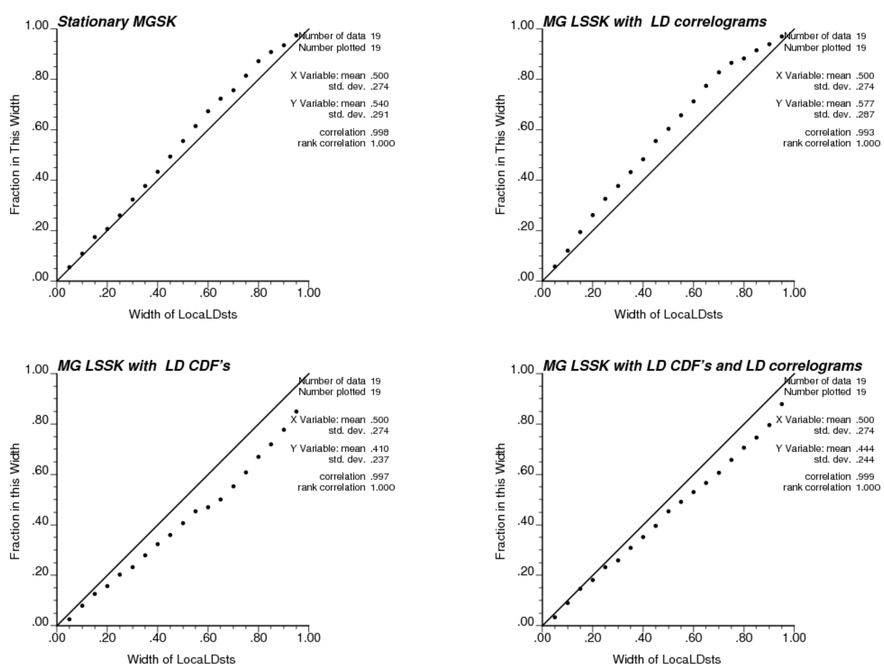

Figure 7: Accuracy plots before postprocessing for multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the clustered dataset.

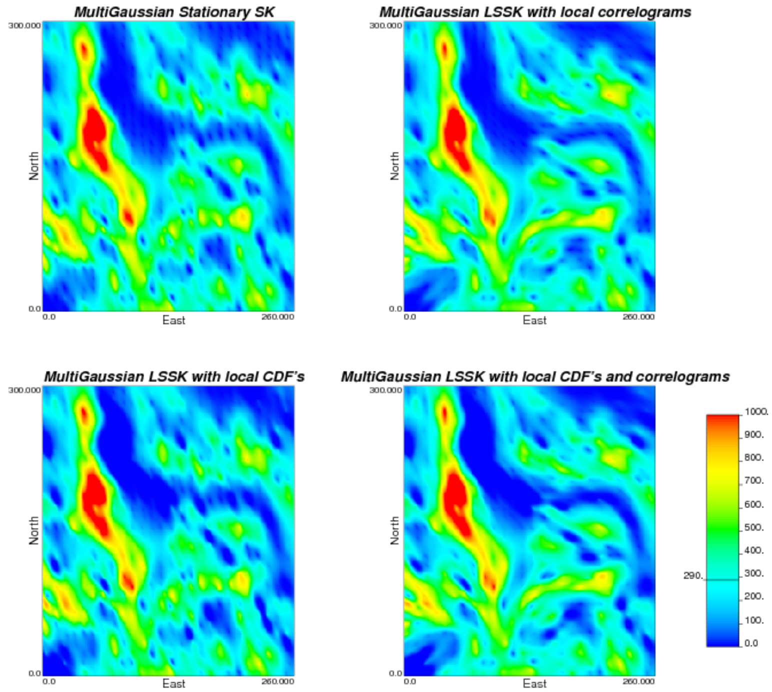

Figure 8: Postprocessed local means obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the gridded dataset and exponential correlogram models.

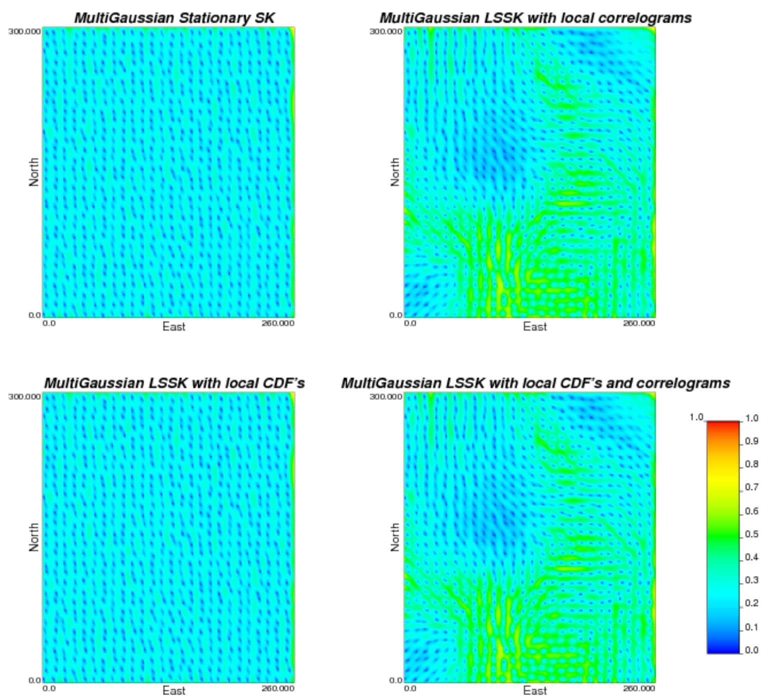

Figure 9: Estimation variances in normal units obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the gridded dataset and exponential correlogram models.

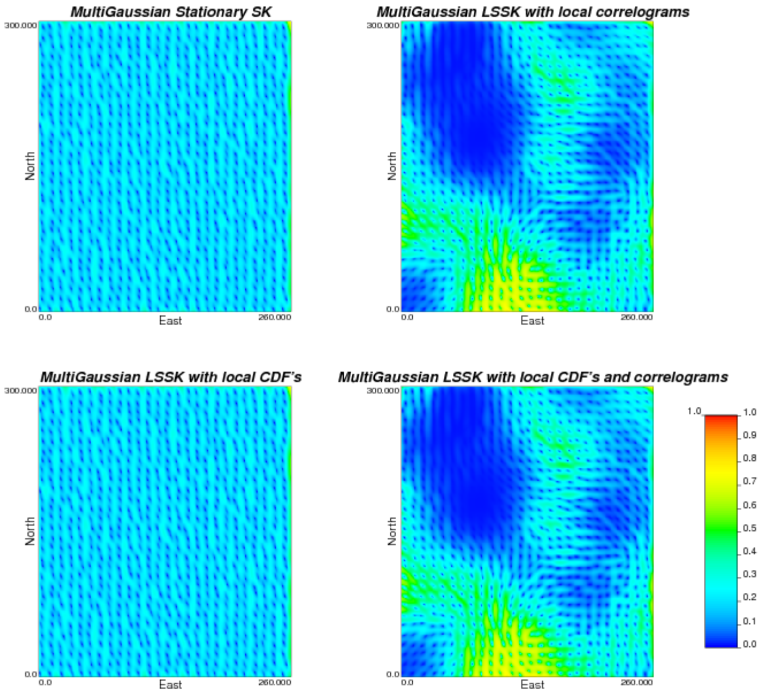

Figure 10: Estimation variances in normal units obtained from multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the gridded dataset and stable correlogram models.

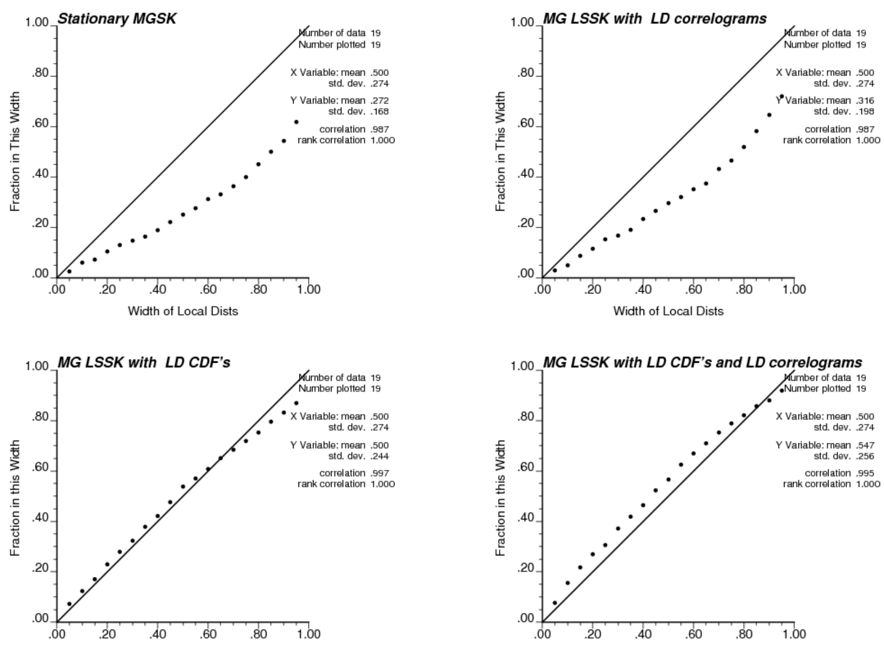

Figure 11: Accuracy plots before postprocessing for multigaussian stationary SK (top left), multigaussian LSSK with location-dependent correlograms (top right), multigaussian LSSK with prior local cdf's (bottom left) and multigaussian LSSK with prior local cdf's and location-dependent correlograms (bottom right) using the gridded dataset and exponential correlogram models.