Statistical Approach to Inverse Distance Interpolation

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Inverse distance interpolation is a robust and widely used estimation technique. Variants of kriging are often proposed as statistical techniques with superior mathematical properties such as minimum error variance; however, the robustness and simplicity of inverse distance interpolation motivate its continued use. This paper presents an approach to integrate statistical controls such as minimum error variance into inverse distance interpolation. The optimal exponent and number of data may be calculated globally or locally. Measures of uncertainty and local smoothness may be derived from inverse distance estimates.

Introduction

Spatial prediction techniques, also known as spatial interpolation techniques, differ from classical modeling approaches in that they incorporate information on the geographic position of the sample data points (Journel and Huijbregts, 1978; Cressie, 1993). Spatial predictions offer means of describing a variety of responses over different spatial scales (Schloeder, et al., 2001). They provide a unique and smooth property distribution that honours the sample points (conditioning data); spatial prediction techniques aim at the local accuracy of resulting uncertainty distributions (Isaaks and Srivastava, 1989; Journel et al., 2000). The most common interpolation techniques calculate the estimates for a property at any given location by averaging nearby data. Weighting for each averaged data value is assigned either according to deterministic or statistical (spatial covariance) criteria. When a statistical criterion is used, the field is considered as a random process and the optimality of the averaging method is determined in terms of minimizing the estimation variance. When a deterministic criterion is used, the measures of optimality are arbitrarily chosen (Borga and Vizzaccaro, 1997). Among statistical methods, geostatistical kriging-based techniques, including Simple and Ordinary Kriging, Universal Kriging and Simple Cokriging (see Journel, 1986; Cressie, 1993; Deutsch, 2002) have been often used for spatial analysis. Among deterministic methods, Inverse Distance Weighted interpolation and its modifications (see Franke, 1982; Nader and Wein, 1998) are the most often applied.

In this paper, we expand the applicability of the inverse distance by introducing a statistical formalism for this method. The proposed formalism is based on the assumption of stationarity and is aimed at providing the estimation variance at the unsampled locations as a measure of accuracy in the inverse distance.

Moreover, based on the derived statistical formalism we propose a general approach to find the optimal exponent value and the optimal number of neighboring points to be used in the inverse distance estimation. It has been noted by many practitioners of the inverse distance weighted interpolation that ID approach is very sensitive to the number of data used in interpolation and to the exponent value; and a significant improvement in estimation precision can be achieved by selecting an optimal number of the closest neighboring points and an optimal exponent value (Kravchenko et al., 1999). Presently, however, there is no exact recommendation about the choice of exponent value and the optimal number of neighboring points to be used in the inverse distance estimation. A number of researchers approached this problem and their recommendations are contradictory. For example, in the case of the inverse distance squared interpolation Morrison (1974), MacDougall (1976), Peucker (1980), and Hodgson (1992) recommended to use respectively, $3 \le k \le 7$, $6 \le k \le 9$, $k \le 6$, and $4 \le k \le 7$ data; however, Declercq (1996), recommended $4 \le k \le 8$ for “smooth” surfaces and $16 \le k \le 24$ for abruptly changing surfaces. In this paper we attempt to explain and document the sensitivity of the inverse distance estimation to the number of data used in estimation and to power used in interpolation. As direct result of the sensitivity analysis, a local inverse distance interpolation approach is proposed to create estimates with minimum achievable estimation variance for the inverse distance interpolation.

Background: Kriging versus Inverse Distance Interpolation

Kriging is a well-proven technique that provides the best linear unbiased estimate and its variance at the unknown location. It is an exact interpolator in the sense that the estimation at a data location returns the original data value. In theory, kriging is a statistically optimal interpolator in the sense that it minimizes estimation variance when the variogram (measure of spatial continuity of the variable under study) is known and under the assumption of stationarity.

Inverse distance weighting estimates the variable of interest by assigning more weight to closer points. It is a simple technique that does not require prior information to be applied to spatial prediction. Despite this simplicity, inverse-distance estimators are shown (experimentally) to be quite sensitive to the type of database, to the number of neighbors used in the estimate, and to the exponent of distance used in weighting (Weber and Englund, 1994). In practical applications, inverse distance weighted interpolation may be preferred over kriging-based techniques when there is a problem of making meaningful estimates of the field spatial structure from sparse data. (Duchon, 1976; Wahba, 1990; Hutchinson, 1993). It is also used when a quick visualisation of the variable under study is required (Borga and Vizzaccaro, 1997). Moreover, a large number of comparative studies among different interpolators based on realistic or geologically sound visual appearance; cross validation and jackknife, which involves consecutively removing a data value from the sample data set and interpolating to that site using the remaining conditioning data values, then comparing the estimated values against the true data (Isaaks and Srivastava, 1989); robustness; or measures of response variables derived from the interpolated property, found that depending upon the situation at hand, inverse distance weighting can be as good or better than geostatistical kriging-based techniques (Weber and Englund, 1992; Gallichand and Marcotte, 1993; Dingman, 1994; Boman et al., 1995; Brus et. al., 1996; Declercq, 1996; Dirks et al., 1998; Moyeed and Papritz, 2002). Therefore, it may be important to analyze the inverse distance weighted interpolation approach in greater detail with the aim of improving it.

The main advantages of kriging over inverse distance interpolation are cited as (1) robustness of estimates with respect to the number of data used in estimation, (2) ability to take into account the spatial structure of the data points (anisotropy) and (3) availability of the estimation variance that yields a measure of the accuracy of any single interpolated value. This measure can have a dual role. Firstly, it evaluates the reliability of our estimates. Secondly, it can serve as a guideline to identify the most uncertain areas for further measurements (Rouhani, 1985).

Inverse Distance Interpolation

An inverse distance interpolation is one of the simplest and most popular interpolation techniques. It combines the proximity concept with the gradual change of the trend surface. An inverse distance (ID) weighted interpolation is defined as a spatially weighted average of the sample values within a search neighborhood (Shepard, 1968; Franke, 1982; Diodato and Ceccarelli, 2005). It is calculated as

$$Z^*(\mathbf{u}) = \sum_{i=1}^{n} \lambda_i Z(\mathbf{u}_i), \tag{1}$$

where $\mathbf{u}$ is the estimation location, $\mathbf{u}_i,\ i = 1,\ldots,n,$ are the locations of the sample points within the search neighborhood, $Z^*(\mathbf{u})$ is the inverse distance estimate at the estimation location, n is the number of sample points, $\lambda_i,\ i = 1,\ldots,n,$ are the weights assigned to each sample point, and $Z(\mathbf{u}_i),\ i = 1,\ldots,n,$ are the conditioning data at sample points. The weights are determined as

$$\lambda_i = \frac{\left(\dfrac{1}{d_i^p}\right)}{\sum_{i=1}^{n}\left(\dfrac{1}{d_i^p}\right)}, \quad (i = 1,\ldots,n), \tag{2}$$

where d_i are the Euclidian distances between estimation location and sample points, and exponent p is the power or distance exponent value. Note that the sum of the inverse distance weights $\lambda_i,\ i = 1,\ldots,n,$ is equal to 1, that is,

$$\sum_{i=1}^{n} \lambda_i = 1.$$

The most common value applied for the power *p* is 2; then estimator in (1)-(2) is called inverse squared distance (ISD) interpolator. However, any value for *p* can be chosen. As *p* increases, the interpolated value by inverse distance is assigned the value of the nearest sample point, that is, inverse distance estimate becomes the same as estimate produced by polygonal method. (Diadato and Ceccarelli, 2005).

The advantage of the inverse distance technique is that it can be easily applied in any number of dimensions and provide reasonable estimates. Nowadays several modifications of the inverse distance method are developed including gradient inverse distance interpolation (GIDW) (Nalder and Wein, 2000).

Statistical Formalism

The uncertainty in the true value at an usampled location $z(\mathbf{u}) \in A$ can be modeled using cumulative probability distribution function of a random variable $Z(\mathbf{u})$,

$$F(\mathbf{u}; z) = \text{Prob}\{Z(\mathbf{u}) \leq z\}.$$

This probability distribution function can be thought of as being a model of the lack of knowledge about the value of the variable under study at the usampled location **u**. Repetitive samples are needed to infer any statistic. Unfortunately, in the spatial context repetitive samples are not available. A measurement cannot be repeated at the same location **u** to obtain probability distribution of the random variable $Z(\mathbf{u})$. Stationarity is a decision to take samples at other locations to obtain a model of the probability distribution. This amounts to assume the invariance of the random function and all its moments by translation over the domain *A*. The first order of stationarity assumes that the mean of the variable of interest is constant throughout the domain *A*; the second order of stationarity assumes that the variance of data is constant throughout the study domain *A* (Deutsch, 2002). That is,

$$\begin{aligned} \mathrm{E}(Z(\mathbf{u})) &= m, \quad \forall \mathbf{u} \in A; \\ \mathrm{Var}(Z(\mathbf{u})) &= \mathrm{E}(Z(\mathbf{u}) - m(\mathbf{u}))^2 = \sigma^2, \quad \forall \mathbf{u} \in A. \end{aligned} \tag{3}$$

The mean and variance of the inverse distance estimator $Z^*(\mathbf{u})$ at estimation location **u** given by (1)-(2) can be derived under the assumption of stationarity as follows

$$\begin{aligned} \mathrm{E}(Z^*(\mathbf{u})) &= \mathrm{E}\left(\sum_{i=1}^{n} \lambda_i Z(\mathbf{u}_i)\right) = \sum_{i=1}^{n} \lambda_i \mathrm{E}(Z(\mathbf{u}_i)) = m \sum_{i=1}^{n} \lambda_i = m; \\ \mathrm{Var}(Z^*(\mathbf{u})) &= \mathrm{Var}\left(\sum_{i=1}^{n} \lambda_i Z(\mathbf{u}_i)\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \mathrm{Cov}(Z(\mathbf{u}_i), Z(\mathbf{u}_j)); \end{aligned} \tag{4}$$

where $\mathrm{Cov}(Z(\mathbf{u}_i), Z(\mathbf{u}_j)), \quad i, j = 1, \cdots, n,$ denotes data-to-data covariance function calculated under assumption of stationarity though the semivariogram model $2\gamma(\mathrm{h})$ (Journel and Huijbregts, 1978).

The estimate and variance of the inverse distance estimator at the data location are set to the data value at that location and stationary domain variance σ^2, respectively. Note, however, that despite neither the IDW estimate and variance at the data location are defined; it can be shown that they converge in a limit to the data value at that precise location and stationary domain variance σ^2, respectively.

Under assumption of stationarity, the variance of the variable under study at each location of the domain should be exactly equal to the stationary domain variance σ^2. However, the map of the inverse distance estimates is smooth. The smoothing effect of inverse distance interpolation technique is directly related to the IDW variance via this expression

$$\text{Smoothing effect} = \sigma^2 - \mathrm{Var}(Z^*(\mathbf{u})) = \sigma^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \mathrm{Cov}(Z(\mathbf{u}_i), Z(\mathbf{u}_j)). \tag{5}$$

Note that smoothing effect of the inverse distance interpolator in (5) can be also referred to as the *missing variance*. This is because by adding the variable with variance (5) to the inverse distance estimate we will obtain new variable with variance equal to the stationary domain variance σ^2. Note that at the data locations missing variance is equal to zero.

Moreover, note that the estimation variance of the inverse distance estimator at the estimation location **u** (under the assumption of stationarity) can be calculated as (Deutsch, 2002)

$$\sigma_{est}^2 = \mathrm{E}[Z - Z^*(\mathbf{u})]^2 = \sigma^2 - 2\sum_{i=1}^{n} \lambda_i \mathrm{Cov}(Z(\mathbf{u}_j), Z(\mathbf{u})) + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \mathrm{Cov}(Z(\mathbf{u}_i), Z(\mathbf{u}_j)). \quad (6)$$

Sensitivity of the Inverse Distance Weighted Interpolation to the Number of Data: Example

There are a total of 310 samples within a 2D rectangular project area extending 3km in the Easting X direction and 5km in the Northing Y direction. The location map of the normal score transformed data together with its histogram is given in Figure 1. The variogram of normal score transformed data is isotropic spherical with nugget effect of zero and range of correlation 1450 meters.

Figure 2 shows results of the inverse distance interpolation for the mean and variance (that is, missing variance) of the local conditional distributions obtained based on 3 data with several exponent values, that is, $p = 1,2,3,9.$ Figure 3 shows analogous results of the inverse distance interpolation but obtained based on 24 data with different exponent values. It can be clearly noted from Figures 2-3 that

- the inverse distance estimates (means of local distributions) becomes smoother with increase in the number of data for the same exponent value; there appears clear pixilation of the higher/lower values. With increase in the number of data and smoothness of the map, the variance of the local conditional distributions increases;
- the map of the estimates (means of local distributions) starts to take on mosaic type appearance with increase in the exponent value for the same number of data, (Chiles and Delfiner, 1999), that is, there is clear boundaries between higher/lower values; the impact of the number of data becomes less important with increase in the exponent value. Also we can note that with increase in the power exponent the variance of the local conditional distributions decreases, the variance of the estimated values approaches stationary domain variance everywhere except at the boundaries between higher/lower values.

To analyze results of estimation in greater detail two slices at X = 100 and at X = 300 are selected. The estimation variances for the inverse distance interpolator with exponent value of 1 (p = 1) as a functions of the number of data for two chosen slices are shown in Figure 4. For comparison, Figure 4 also shows the estimation variances for the ordinary kriging interpolator as a function of the number of data for the same two slices.

Looking at Figure 4 we can clearly note that with increase in the number of data for the ordinary kriging interpolator there is only minor change in the estimation variance (that is, estimation variance slightly decreases with increase in the number of data). However, contrary to ordinary kriging, with increase in the number of data for the inverse distance interpolator there is generally an increase in the estimation variance. Note that the increase in the inverse distance estimation variance is quite substantial when using 24 data instead of 3. Looking at Figure 4 we can also conclude that in order to minimize the estimation variance, a small number of values (3-6 data) should be used for estimation. Ordinary kriging estimation variance will of course always be smaller than that of the inverse distance, however for small number of data used in the inverse distance interpolation this difference is quite small, see Figure 4.

The estimation variances for the inverse distance interpolator obtained based on 3 and 24 data as a functions of the exponent value for two slices two slices at the X = 100 and at the X = 300 are shown in Figure 5. Note that when smaller number of data is used for interpolation, the difference in the estimation variance is minimal. Large power exponents (p = 9) produce estimates with larger estimation variance. On the other hand, when larger number of data is used for interpolation, the difference in the estimation variance for different exponent values is more pronounced. The inverse distance interpolation with higher exponent value is producing better result. In fact, it is interesting to note that when large exponent value is used for interpolation, like p = 9, number of data used does not have any effect on the results, see Figure 6.

Sensitivity of the variance of the inverse distance interpolator to the number of data and power exponent is shown in Figures 7 and 8. Note from these figures the strong change in the variance of the inverse distance interpolator with change in the parameters opposed to the inverse distance interpolator.

Inverse Distance with Locally Varying Parameters

The optimal inverse distance weighted interpolation parameters, that is, number of data and power exponent, can be chosen by minimizing estimation variance at each location. Depending on the estimation location, of course, optimal parameters will be different as well as will be different estimation variance. Estimation variance obtained is the minimum estimation variance achievable by the inverse distance interpolation technique. Figure 9 shows the result of the optimal local inverse distance interpolation for the mean and variance of the local conditional distributions. The following values for the power exponent were considered: p = 1 to p = 12 with step 0.5; the following values for the number were considered: N = 3 to N = 31 with step 2. Figure 9 also shows the map of the optimum number of data and exponent power for all estimation locations in the study domain. On average, over the study domain the average number of data used in estimation is 5.3 and an average power exponent is 1.93. However, for some locations the power exponent was as high as 12 while for others as low as 1, the same applies to the number of data.

Table 1 shows results of the cross validation for all 310 data in the study domain obtained based on the inverse distance interpolation with 3, 6, 12 and 24 data and power exponent p = 1, 2, 3, 4 and 6. Table 1 also shows results of the cross validation obtained based on the local inverse distance interpolation. Clearly, local inverse distance interpolation outperforms the inverse distance interpolation with constant parameters.

Conclusions and Discussion

There are several reasons why inverse distance interpolation may be preferred over the geostatistical kriging-based techniques. Besides the fact that it is simple, applicable to any number of dimensions, it is also robust in estimation, does not suffer from the string effect of kriging (Deutsch, 1993 1994); does not result in negative weights – no screening effect (Deutsch and Journel, 1998); and does not require solving system of equations for the weights. Moreover, it provides reasonable estimates and is shown in a large number of comparative studies to be even better than geostatistical kriging-based techniques (Weber and Englund, 1992). A statistical formalism is proposed for the deterministic inverse distance. This formalism derived based on the assumption of stationarity and a known variogram model allowed us not only to derive the variance of the inverse distance estimates and the variance of the local conditional distributions as measure of the accuracy of any single interpolated value. A general procedure was developed for selecting the optimal number of data and exponent value for the inverse distance estimation of each location separately in the study domain. The developed procedure, referred to as the local inverse distance interpolation is shown to performed better than inverse distance interpolation with fixed parameters for several different sets of parameters in cross validation.

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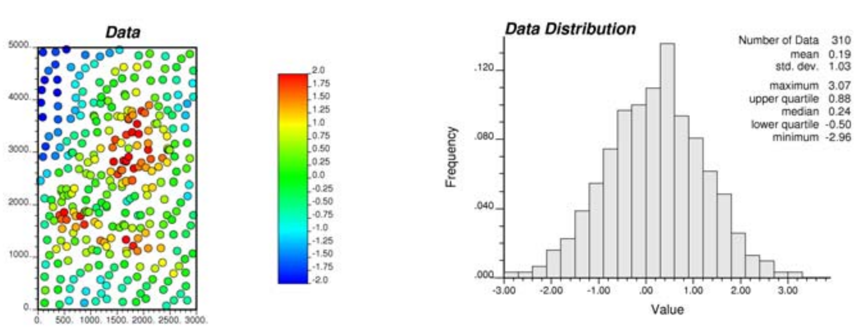
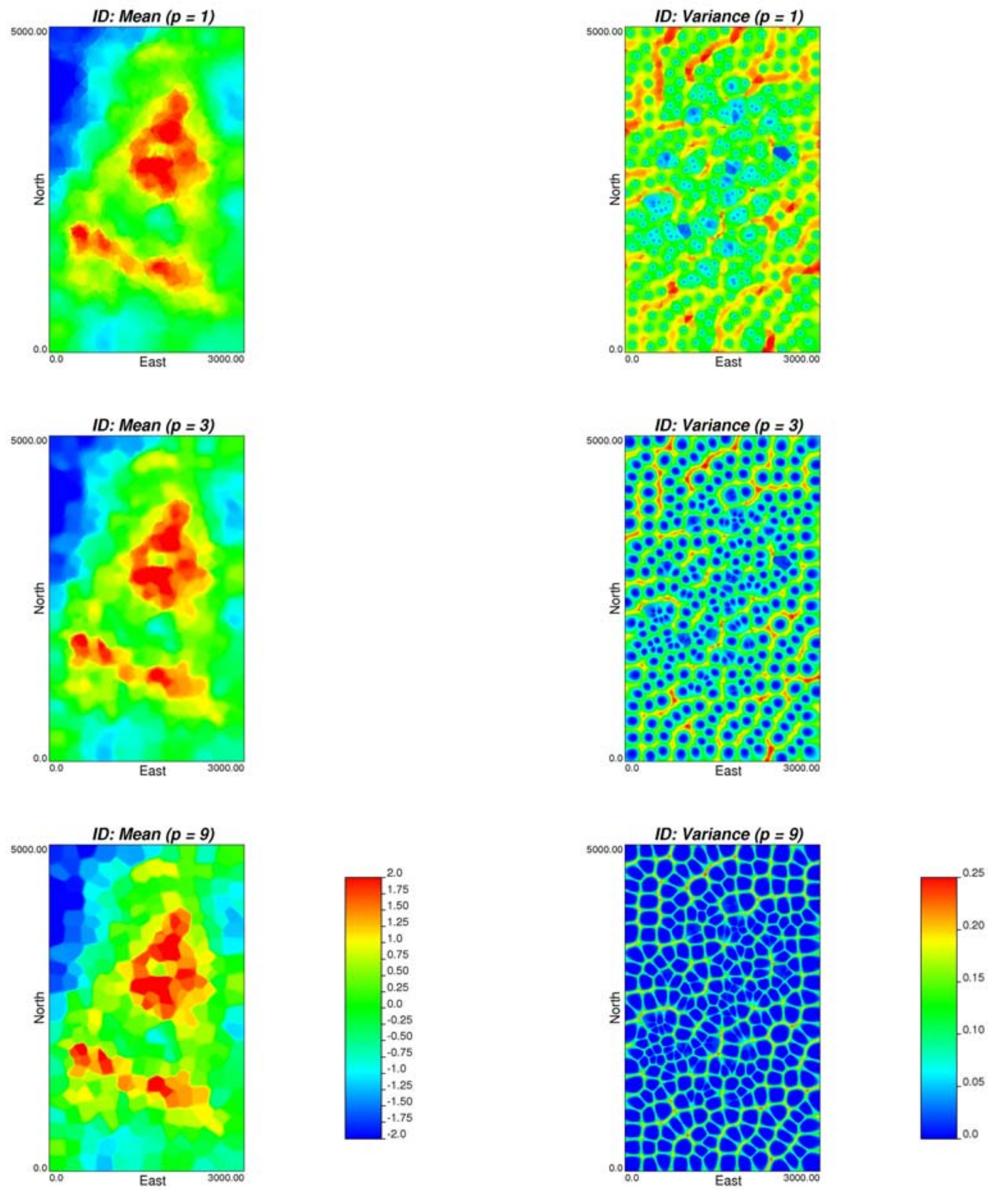
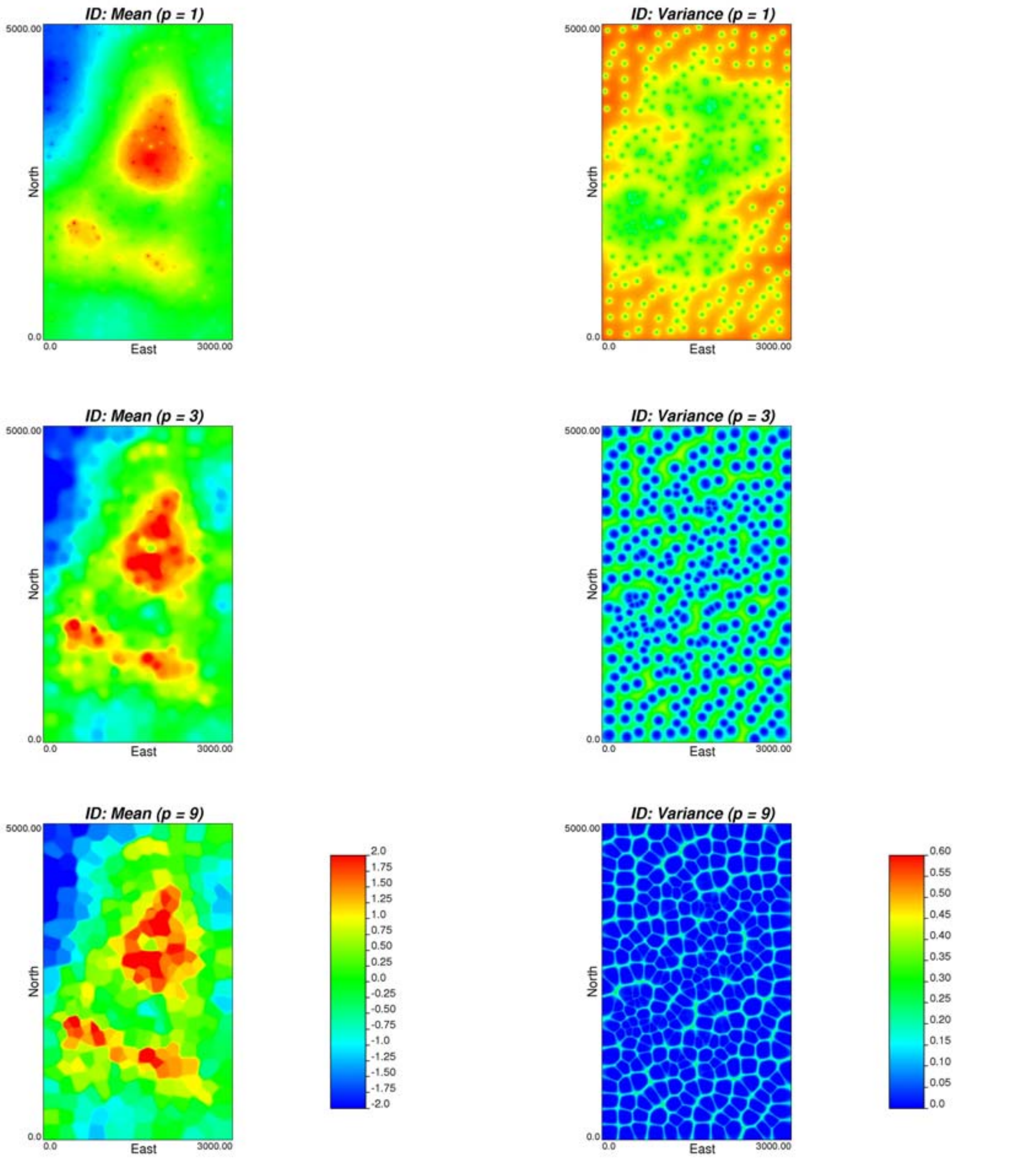


Figure 1: Location map of 310 samples (a) together with their distribution (b).



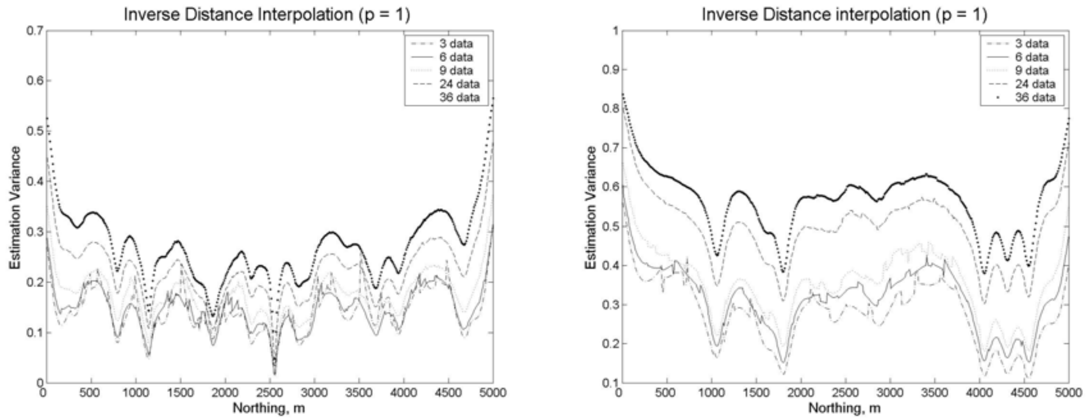
a) b)

Figure 2: Results of the inverse distance interpolation for the mean (a) and variance (b) of the local conditional distributions obtained based on 3 data with exponent value p equal to: 1 (top); 3 (middle) and 9 (bottom).

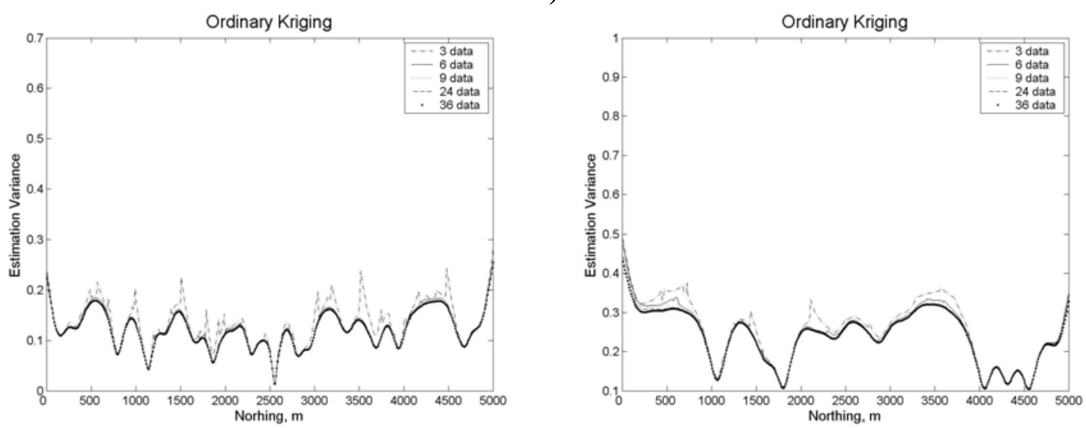


a) b)

Figure 3: Results of the inverse distance interpolation for the mean (a) and variance (b) of the local conditional distributions obtained based on 24 data with exponent value p equal to: 1 (top); 3 (middle) and 9 (bottom).



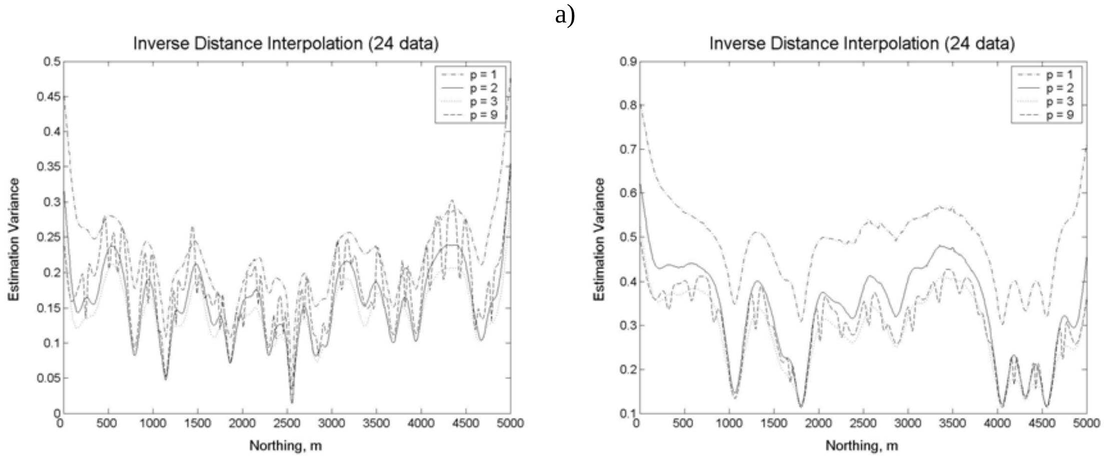
a)



b)

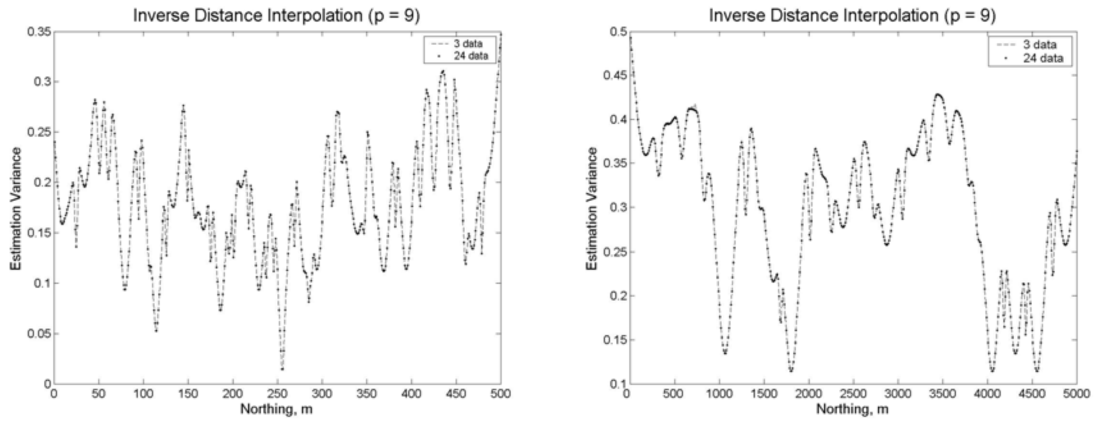
Figure 4: The estimation variances for the inverse distance interpolator with exponent value of 1 (p = 1) (a) and the estimation variances for as the ordinary kriging estimator (b) as a function of the number of data for slice at X = 100 (left) and at X = 300 (right).

a)



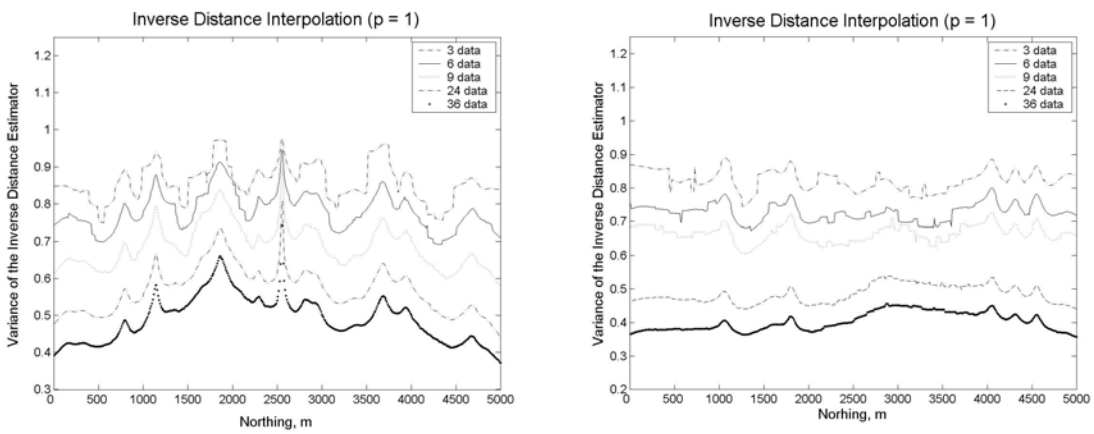
b)

Figure 5: The estimation variances for the inverse distance interpolator obtained based on 3 data (a) and the estimation variances for the inverse distance interpolator obtained based on 24 data (b) as a function of the power exponent for slice at X = 100 (left) and at X = 300 (right).



a) b)

Figure 6: The estimation variances for the inverse distance interpolator with exponent value of 9 (p = 9) as a function of the number of data for slice at X = 100 (a) and at X = 300 (b).



a) b)

Figure 7: The variances for the inverse distance interpolator with exponent value of 1 (p = 1) as a function of the number of data for slice at X = 100 (a) and at X = 300 (b).

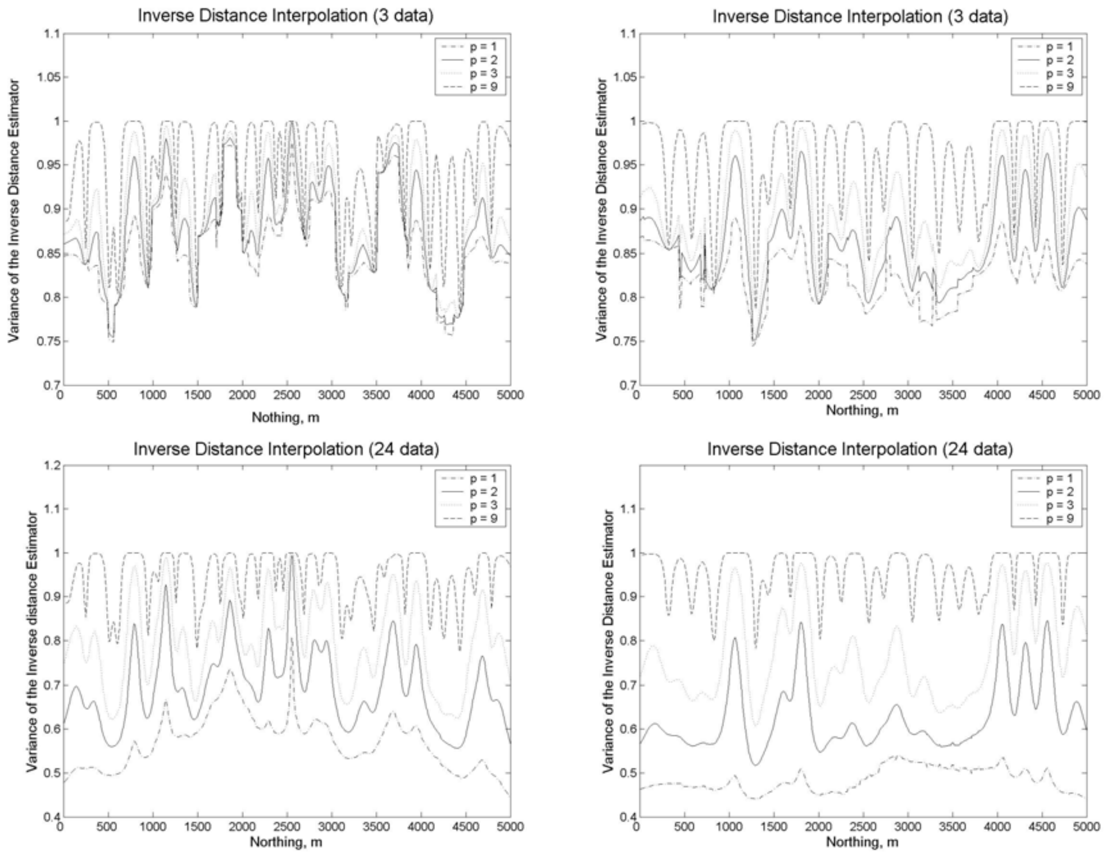


Figure 8: The variances for the inverse distance interpolator obtained based on 3 data (a) and 24 data (b) as a function of the power exponent for slice at $X = 100$ (left) and at $X = 300$ (right).

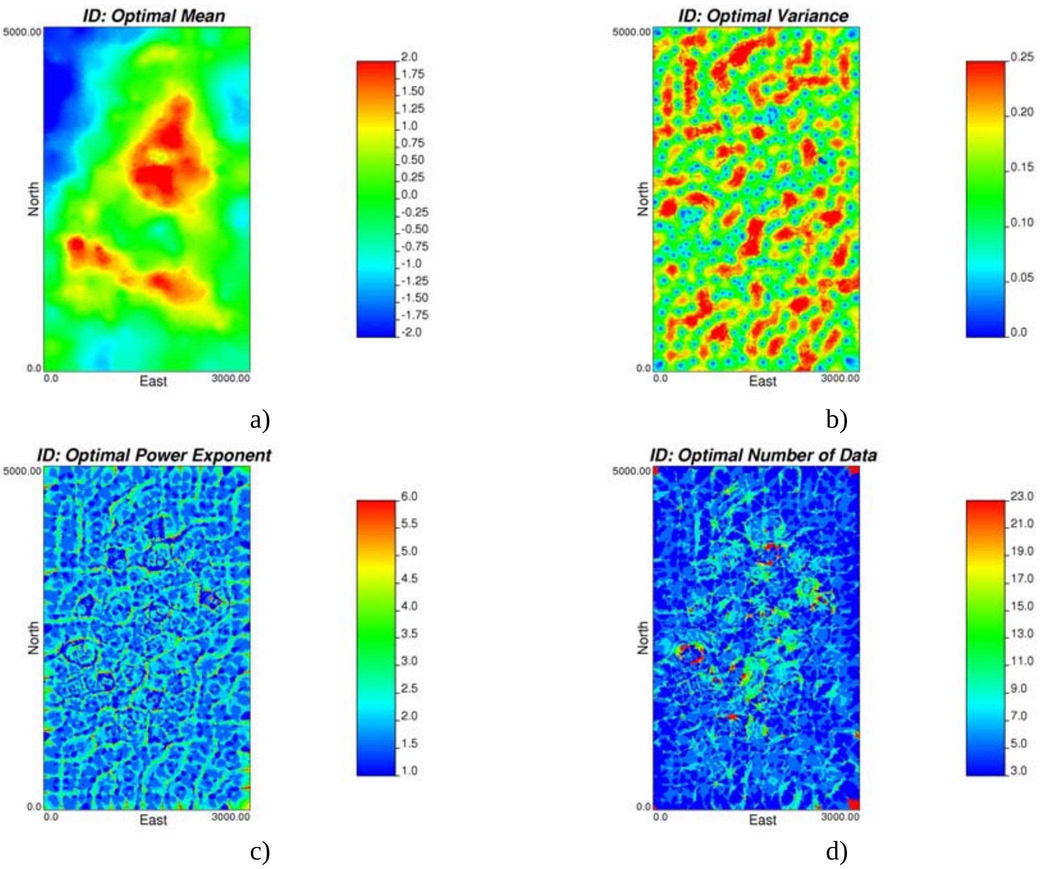


Figure 9: Result of the optimal local inverse distance interpolation for the mean (a) and variance (b) of the local conditional distributions; optimal power exponent (c) and optimal number of data (d) for all estimation locations in the study domain. (*Available in black and white*)

Table 1: Results of the cross validation for all 310 data in the study domain obtained based on the inverse distance interpolation with 3, 6, 12 and 24 data and power exponent p = 1, 2, 3, 4 and 6 and based on the local inverse distance interpolation with optimal parameters.

Number of Data	P = 1	P = 2	P = 3	P = 4	P = 6
3 data	0.1394	0.1339	0.1326	0.1338	0.1394
6 data	0.1676	0.1464	0.1364	0.1330	0.1359
12 data	0.1965	0.1618	0.1428	0.1351	0.1358
24 data	0.2542	0.1889	0.1529	0.1384	0.1361
Optimal parameters	0.1315				