A New Model of Coregionalization for Variograms of Different Shape

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A common challenge for multivariate geostatistical analyses is modeling variables that have been measured on different supports. Variograms and cross variograms for these variables will often exhibit different shapes; the variogram of larger support data showing more short scale continuity. The linear model of coregionalization (LMC) cannot be used in these cases. A new coregionalization model for variograms of different shape is required. An extended linear model of coregionalization (ELMC) is presented here that modifies the underlying covariance functions with a volume regularization operator. The resulting model can be used to calculate licit, semi-positive definite covariance models suitable for modeling variograms of different shape. The ELMC is implemented in the program `vmodel_elmc`.

Introduction

The linear model of coregionalization (LMC) remains one of the only practical multivariate coregionalization models available to the geostatistician in all but the most trivial of cases. There are, however, a number of drawbacks to the LMC including difficulties in constructing semi-positive definite models and the inability to deal with variograms or covariance functions with different shapes. The construction of licit models can be addressed by using semi-automatic fitting software, such as `varfit_lmc`; however all variogram models must share the same structure shapes (ie: spherical, exponential, Gaussian). This limitation is significant when variables are measured on different volume supports.

When correlated variables are measured on different volume supports, as would be the case for core data and remote sensing, the variograms will often have different shapes. Remote sensing data may exhibit a Gaussian shape due to the large support volume compared to a spherical variogram, often with a significant nugget effect, observed for core data. The cross variogram or covariance will then generally exhibit a Gaussian shape. This situation is shown in Figure 1. For this situation it is impossible to model both the short scale variability of the spherical Z1 and Gaussian Z1Z2 and Z2 accurately using the LMC.

Recent work by Marcotte (2012) on a generalized linear coregionalization model (GLCM) is very promising to address this issue. Marcotte presents 3 operators which modify the implicit underlying variables in the LMC. The observed random variables are then given as a sum of underlying random variables and functions of these variables. These operators are the first order partial derivative, a location shift and a regularization operation. Of these operators, the regularization operator is of the most interest to the authors because it has the potential to address the problems presented by variograms of different shape. In this paper, an extended linear model of coregionalization (ELMC) is introduced based on the ideas of Marcotte's regularization operator. The ELMC incorporates the underlying variables of the LMC with weighted volume regularization functions of these variables. A program, `vmodel_elmc` is presented which implements the ELMC for plotting covariance functions.

Theory

Recall the definition of the linear model of coregionalization; notation used here will be similar to that used by Marcotte (2012). A K x 1 random vector of observed variables, **Z**, is taken to be a linear combination of M orthogonal, underlying (unobserved) variables **Y**. The decision of stationarity for **Z** for geostatistical and covariance modeling also applies to the **Y** underlying random variables. We chose that the **Y** variables are standardized (mean of zero, variance of 1) and are related to **Z** by a K x M matrix of unknown coefficients, **A** as:

$$\mathbf{Z} = \mathbf{A}\mathbf{Y} \tag{1}$$

If the mean of **Z** is non-zero, then a mean vector could also be added without affecting any of the covariance relationships developed in this paper. The orthogonality constraint on **Y** implies that:

$$\mathrm{Cov}\{Y_i(\mathbf{u}), Y_j(\mathbf{u}+\mathbf{h})\} = 0 \ \forall \mathbf{h} \text{ where } i \neq j \ \forall \ i, j \text{ from } 1,...,M \tag{2}$$

The covariance matrix of **Z** is then:

$$\mathbf{C_Z} = \mathbf{A}\mathbf{C_Y}\mathbf{A}' \tag{3}$$

The linear model of coregionalization (LMC) is given by covariance components $\mathbf{C}_l(\mathbf{h})$ with contributions described by K x K $\mathbf{B}_i$ semi-positive definite matrices:

$$\mathbf{C_Z}(\mathbf{h}) = \sum_{l=1}^{L} \mathbf{B}_l \mathbf{C}_l(\mathbf{h}) \tag{4}$$

As noted by Marcotte, the $\mathbf{B}_i$ matrices are from the diagonal of **AA'**. For L structural components the dimension of **Y** is implicitly specified to be $M = K \times L$. Generally fitting programs like `varfit_lmc` only fit the $\mathbf{B}_i$ matrices. The **A** matrices are assumed implicitly by the model. Marcotte considered the addition of functions of **Y** to the **Z** variable to form a generalized linear coregionalization model. For this extended linear model of coregionalization, we restrict our interest to consider a weighted, volume (v) averaged operator to address challenges caused by different variogram shapes. Consider the variable $\overline{Y_m}$ to be the weighted, volume averaged value of the variable Y_m. The weight function $w_m(\mathbf{u})$ is taken to be a radial basis function around the location **u**.

$$\overline{Y}_m(\mathbf{u}) = \frac{1}{v}\int_v w_m(\mathbf{u}) Y_m(\mathbf{u}) d\mathbf{u} \tag{5}$$

The extended model for **Z** can then be written as:

$$\mathbf{Z} = \mathbf{A}\mathbf{Y} + \overline{\mathbf{A}}\overline{\mathbf{Y}} \tag{6}$$

The resulting covariance matrix is a series of matrix multiplications on 4 covariance matrices (see Appendix for derivation):

$$\mathbf{C_Z} = \mathbf{A}\mathbf{C_Y}\mathbf{A}' + \mathbf{A}\mathbf{C}_{\mathbf{Y}\overline{\mathbf{Y}}}\overline{\mathbf{A}}' + \overline{\mathbf{A}}\mathbf{C}_{\overline{\mathbf{Y}}\mathbf{Y}}\mathbf{A}' + \overline{\mathbf{A}}\mathbf{C}_{\overline{\mathbf{Y}}}\overline{\mathbf{A}}' \tag{7}$$

The covariance matrix for **Z** is symmetric and requires 3 new covariance types to be calculated. The cross covariances between different volume averaged variables are zero; these degenerate as the original variables are orthogonal. The three new required covariances are therefore:

$$1)\ \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u})\} = \mathrm{Var}\{\overline{Y}_m(\mathbf{u})\}$$

$$2)\ \mathrm{Cov}\{Y_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\} = \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), Y_m(\mathbf{u}+\mathbf{h})\}$$

$$3)\ \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\}$$

The integrals for the three types of covariance functions involving the volume averaged underlying variables can be calculated from (5) (see Appendix for derivation):

$$1)\ \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u})\} = \mathrm{Var}\{\overline{Y}_m(\mathbf{u})\} = \frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1) w_m(\mathbf{u}_2) C_{Y_m}\left(\mathbf{u}_1 - \mathbf{u}_2\right) d\mathbf{u}_1 d\mathbf{u}_2$$

$$2)\ \mathrm{Cov}\{Y_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\} = \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), Y_m(\mathbf{u}+\mathbf{h})\} = \frac{1}{v}\int_v w_m(\mathbf{u}_2+\mathbf{h})\, \mathrm{C}\left(\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{h}\right) d\mathbf{u}_2 \tag{8}$$

$$3)\ \mathrm{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\} = \frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1) w_m(\mathbf{u}_2+\mathbf{h})\, \mathrm{C}\left\{\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{h}\right\} d\mathbf{u}_1 d\mathbf{u}_2$$

All three covariance functions can only be expressed symbolically as far as the volume integrals without analytical functions for the volume, weights and covariance functions. At this point, these covariance functions are theoretically licit for calculating the ELMC. Practical implementation of the ELMC requires restricting ourselves to specific covariance, weight and volume functions. These restrictions will be discussed together with numerical implementation details.

Numerical Implementation

Many of the assumptions made are the same as for the LMC. We restrict ourselves to selected covariance functions valid in 3 dimensions for which the covariance is only a function of an anisotropic lag separation vector **h**. Possible covariance functions could include the nugget effect, spherical, exponential or Gaussian covariance functions as implemented in standard GSLIB (Deutsch and Journel, 1998) and CCG programs.

The volume integrals can be approximated by numerical integration techniques. Regularization volumes we consider are rectangular prisms and ellipsoids. We consider the weighting functions to be anisotropic radial basis functions over an arbitrary ellipsoid. The radial basis functions considered are (Figure 2):

$$\begin{aligned}&\text{1) Equal weighted: } w_i(\mathbf{u}) = c\\&\text{2) Linear: } w_i(\mathbf{u}) = 1 - c \cdot r(\mathbf{u})\\&\text{3) Inverse multiquadric: } w_i(\mathbf{u}) = \frac{1}{\sqrt{1+(c \cdot r(\mathbf{u}))^2}}\end{aligned} \tag{9}$$

Where r is the anisotropic normalized radius and c is an arbitrary constant. If the location **u** falls outside of the weighting ellipsoid (Figure 2), the weight is null. Assuming that a reasonably large weighting domain which spans the regularization volume is chosen, the weighting function will be continuous and smooth over the domain of interest. In addition, the covariance functions considered are continuous and smooth over the regularization volume. The nugget effect is a special case for which the volume integral is exactly zero, and hence the covariance of the volume averaged variables is zero as well.

For the smooth weighting and covariance functions that will be considered, both of these shapes can be integrated efficiently using numerical quadrature algorithms such as those compiled by Stroud (1971). These algorithms discretize the shape into n points with weights ω_j for j=1,…,n, $\sum\omega_j=1$ for which the integrand is evaluated and summed. For double integrals, this procedure is iterated. The three types of covariance functions in Eq. 8 can then be expressed as:

$$\begin{aligned}&\text{1) } \operatorname{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u})\} = \operatorname{Var}\{\overline{Y}_m(\mathbf{u})\} \approx \frac{1}{v^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\omega_j\omega_k w_i(\mathbf{u}_j) w_i(\mathbf{u}_k) C_{Y_i}\left(\mathbf{u}_j - \mathbf{u}_k\right)\\&\text{2) } \operatorname{Cov}\{Y_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\} = \operatorname{Cov}\{\overline{Y}_m(\mathbf{u}), Y_m(\mathbf{u}+\mathbf{h})\} \approx \frac{1}{v}\sum_{j=1}^{n}\omega_j w_i(\mathbf{u}) C_{Y_i}\left(\mathbf{u} - \mathbf{u}_j + \mathbf{h}\right)\\&\text{3) } \operatorname{Cov}\{\overline{Y}_m(\mathbf{u}), \overline{Y}_m(\mathbf{u}+\mathbf{h})\} \approx \frac{1}{v^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\omega_j\omega_k w_i(\mathbf{u}_j) w_i(\mathbf{u}_k+\mathbf{h}) C_{Y_i}\left(\mathbf{u}_j - \mathbf{u}_k + \mathbf{h}\right)\end{aligned} \tag{10}$$

Increased numerical efficiency can be obtained by neglecting the volume averaging in (10); this approach was taken in the numerical implementation used. The two quadrature algorithms chosen were the 13 point C3 5-1 algorithm for a unit cube and the 21 point S3 5-2 algorithm for a sphere (Stroud, 1971). The integration results are exact if the integrand is a polynomial of degree 5 or less. Higher degree formulas could be substituted if a more precise estimate of the integral was required at the expense of increased computational time for all covariance calculations.

For non-unit volumes, the integrand evaluation points and weights must be rescaled. There are two possible approaches to determine the correct coordinates and weights. In the first approach, the full affine transformation could be considered. Details for this transformation applied to numerical quadrature are given by Stroud (1971). A more intuitive approach from the authors' perspective is to divide the problem up into a coordinate rotation and linear scaling. First, the evaluation points are rotated to be along the major directions of anisotropy in the ellipsoid or rectangular prism. These points are then scaled using the anisotropy values in each direction. This two-step procedure can be expressed using a series of simplified affine transformation matrices. Given x,y,z values contained in a unit sphere or cube, the rotation and scaling using a maximum range according to GSLIB (Deutsch and Journel, 1998) conventions is:

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\theta + \sin\alpha\sin\beta\sin\theta & -\sin\alpha\cos\theta + \cos\alpha\sin\beta\sin\theta & -\cos\beta\sin\theta \\ \sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \cos\alpha\sin\theta - \sin\alpha\sin\beta\cos\theta & -\sin\alpha\sin\theta - \cos\alpha\sin\beta\cos\theta & \cos\beta\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} a1_v & 0 & 0 \\ 0 & a2_v & 0 \\ 0 & 0 & a3_v \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$

This transformation is similar to the anisotropy transformation used in variogram computation. The determinant of a diagonal matrix is simply the product of the diagonal entries. The weights are rescaled by the determinant of the diagonal affine transformation matrix and the points by the rotation matrix.

Specification of the Extended Linear Model of Coregionalization

The linear model of coregionalization is specified by the covariance functions for each of the structures for each regionalized variable and the cross covariances between the variables. Specification of the extended linear model of coregionalization differs since it requires the coefficient matrices **A** and $\bar{\mathbf{A}}$, the covariance structures for the underlying variables in addition to the regularization volume and weighting function for each of the underlying variables. The resulting covariance model for **Z** is then fully defined. Calculation and plotting of the ELMC has been implemented in the GSLIB-like `vmodel_elmc`. The parameter file is given in Table 1.

Table 1: Sample difference summary file comparing estimation differences for *red.dat* test case.

Line	Output
1	Parameters for VMODEL_ELMC
2	**************************
3	
4	START OF PARAMETERS:
5	vmodel_elmc.var -file for variogram output
6	2 80 -number of directions and lags
7	0.0 0.0 1.0 -azm, dip, lag distance
8	90.0 0.0 1.0 -azm, dip, lag distance
9	2 3 -kvar (observed), mvar (unobserved)
10	0.05 0.80 0.6 -A matrix coef. (kvar x mvar matrix row k=1)
11	0.0 0.33 0.14 -A matrix coef. (kvar x mvar matrix row k=2)
12	0.0 0.0 0.0 -Abar matrix coef. (kvar x mvar matrix row k=1)
13	0.0 0.5 0.5 -Abar matrix coef. (kvar x mvar matrix row k=2)
14	0 0.0 0.0 0.0 -it,ang1,ang2,ang3 (covariance function m=1)
15	0.0 0.0 0.0 - a_hmax, a_hmin, a_vert (m=1)
16	1 0.0 0.0 0.0 -vt,ang1,ang2,ang3 (volume definition m=1)
17	0.0 0.0 0.0 - v_hmax, v_hmin, v_vert (m=1)
18	1 0.0 0.0 0.0 0.0 -wt,wpar,ang1,ang2,ang3 (weight definition m=1)
19	0.0 0.0 0.0 - w_hmax, w_hmin, w_vert (m=1)
20	1 0.0 0.0 0.0 -it,ang1,ang2,ang3 (covariance function m=2)
21	38.0 38.0 38.0 - a_hmax, a_hmin, a_vert (m=2)
22	1 0.0 0.0 0.0 -vt,ang1,ang2,ang3 (volume definition m=2)
23	4.0 4.0 4.0 - v_hmax, v_hmin, v_vert (m=2)
24	1 1.0 0.0 0.0 0.0 -wt,wpar,ang1,ang2,ang3 (weight definition m=2)
25	10.0 10.0 10.0 - w_hmax, w_hmin, w_vert (m=2)
26	1 0.0 0.0 0.0 -it,ang1,ang2,ang3 (covariance function m=3)
27	100.0 50.0 12.0 - a_hmax, a_hmin, a_vert (m=3)
28	1 0.0 0.0 0.0 -vt,ang1,ang2,ang3 (volume definition m=3)
29	4.0 4.0 4.0 - v_hmax, v_hmin, v_vert (m=3)
30	1 1.0 0.0 0.0 0.0 -wt,wpar,ang1,ang2,ang3 (weight definition m=3)
31	10.0 10.0 10.0 - w_hmax, w_hmin, w_vert (m=3)

Similar to the original `vmodel`, plotting parameters are specified in lines **6-8**. The numbers of observed and unobserved variables are specified in line **9**. The **A** matrix is specified in lines **10-11** and $\bar{\mathbf{A}}$ matrix in lines **12-13**. More lines and values are added in the parameter file to match the M and K values. For m=1, the variogram is specified in lines **14-15**. Specification of the covariance type follows GSLIB (Deutsch and Journel, 1998) conventions, although a structure of 0 is used to specify a nugget effect. Lines **16-17** specify the volume regularization function. The volume type, `vt`, is either 1 (ellipsoid) or 2 (rectangular prism). The rotation angles and ranges of the ellipsoid or rectangular prism are then specified. Lines **18-19** specify the weighting type, `wt`, from (9) where the weighting parameter, `wpar`, is the constant in (9), c. As before, the rotation angles and ranges specify the weighting function ellipsoid. This series of 6 lines is repeated for m=2 (lines **20-25**) and m=3 (lines **26-31**).

A common situation stemming from the use of variables with different support volumes was presented in Figure 1. Using `vmodel_elmc`, these variables were fit (Figure 3). The fit obtained could be improved with the use of automatic fitting software; however over short distances the ELMC used fits the covariance shape (Gaussian or spherical) very well. The parameter file used is included in the accompanying electronic files.

With such a computationally intensive covariance model, verification that computation times are reasonable was necessary. Calculating the full **Z** covariance matrix one million times took 46 minutes in an unoptimized debugging environment. This time could be decreased with compiler and calculation optimization.

Conclusions and Future Work

The ELMC is very promising to address situations where variograms of different shape must be fit. A program, `vmodel_elmc`, has been presented which implements the calculation of covariance functions using a specified ELMC. Future work with the ELMC would involve implementation into kriging or simulation algorithms and testing with real data sets. A semi-automatic fitting program would also be necessary to assist geostatisticians in fitting the large number of matrix coefficients and covariance function parameters.

References

Deutsch, C.V. and Journel, A.G., 1998. GSLIB: Geostatistical Software Library and User's Guide. Applied Geostatistics Series. Oxford University Press, New York, New York, 369 pp.

Marcotte, D., 2012. Revisiting the Linear Model of Coregionalization. In: P. Abrahamsen, R. Hauge and O. Kolbjørnsen (Eds.), Geostatistics Oslo 2012. Quantitative Geology and Geostatistics. Springer Netherlands, pp. 67-78.

Stroud, A.H., 1971. Approximate Calculation of Multiple Integrals. Prentice-Hall Inc., Englewood Cliffs, N.J., 431 pp.

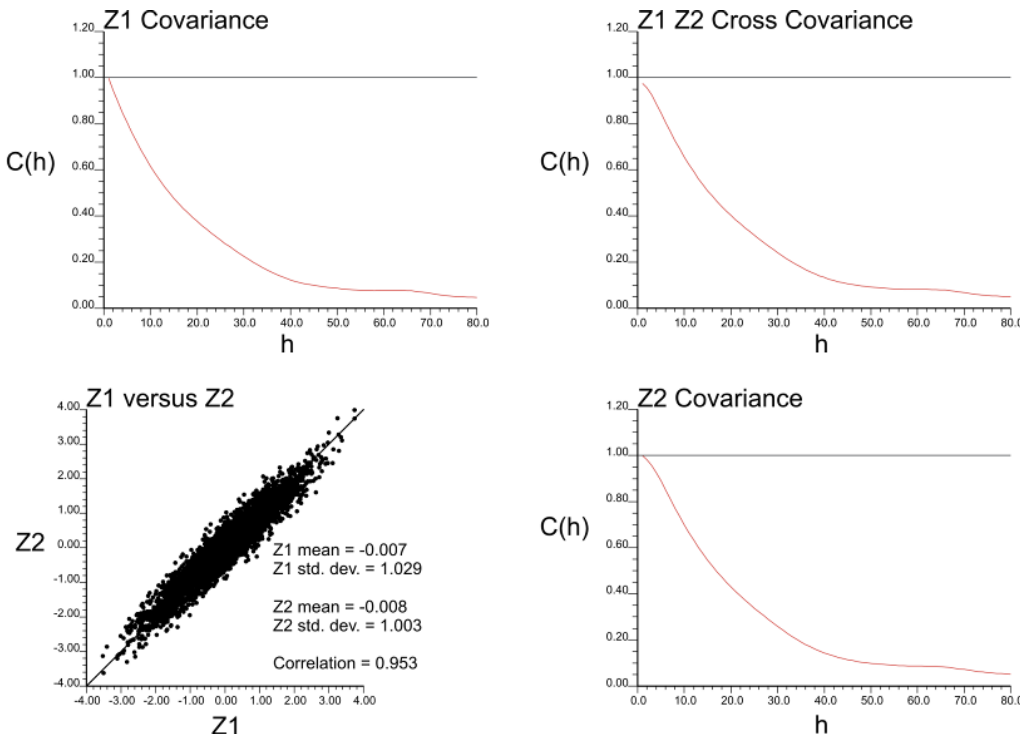

Figure 1: Covariance plots for data from a small support volume (Z1), a highly correlated variable with a large support volume (Z2) and the cross covariance and associated scatterplot.

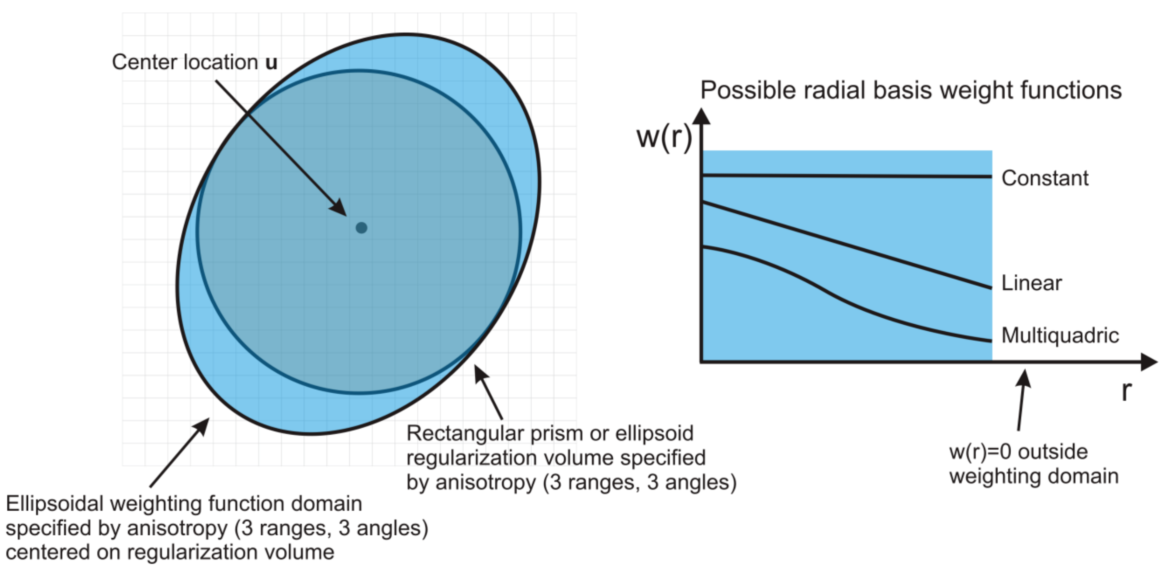

Figure 2: Regularization volume and weighting function domain sketch and illustration of possible radial basis weight functions.

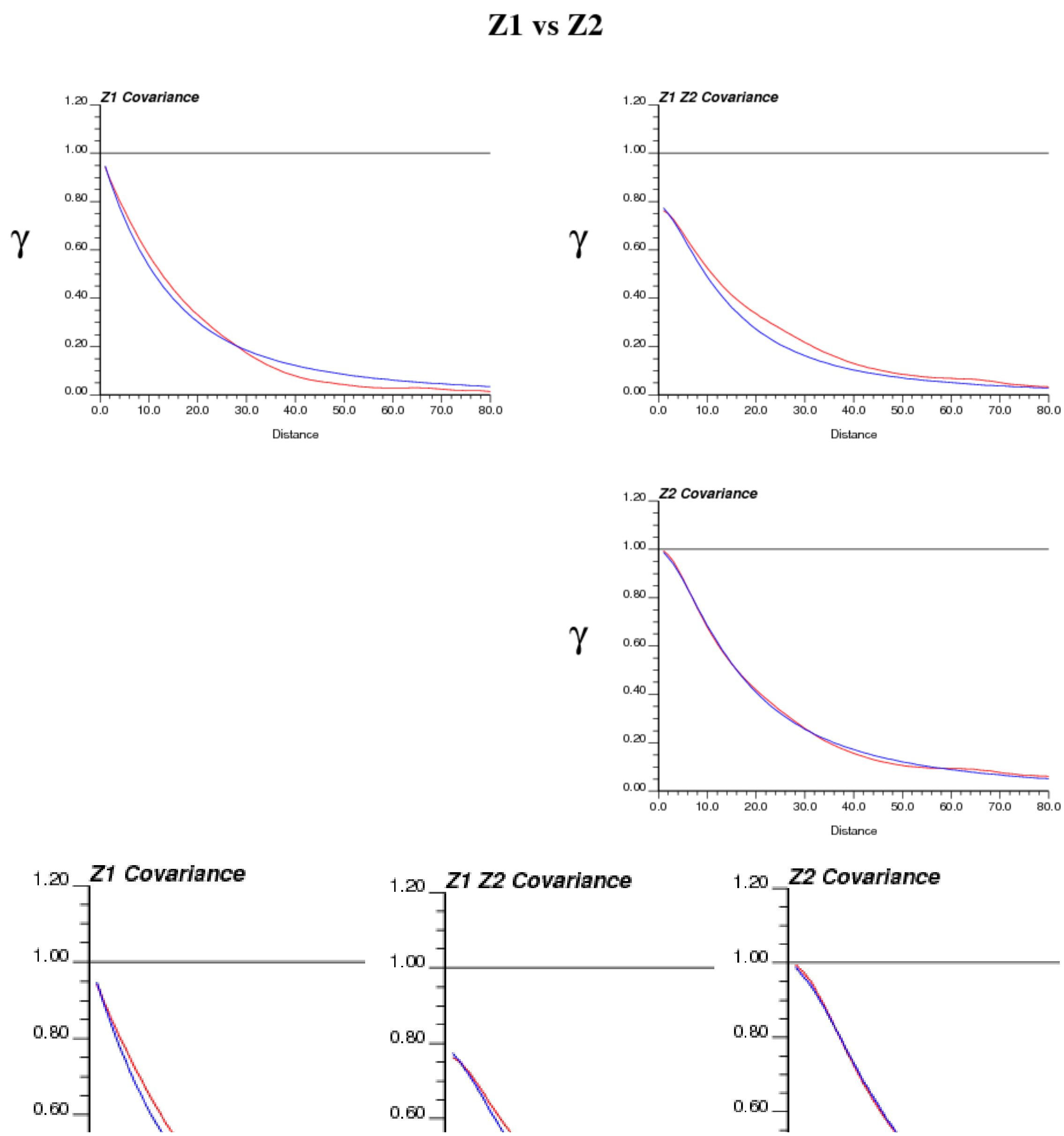

Figure 3: Covariance plots for data (red) from a small support volume (Z1), a highly correlated variable with a large support volume (Z2) and the cross covariance fit with the extended linear model of coregionalization (blue).

Appendix: Mathematical Details

Mathematical details and relevant derivations are presented in this appendix. Additional mathematical details are in the extended electronic appendix.

Derivation of Covariance of Z Variables with the Extended Linear Model of Coregionalization

The covariance matrix of **Z** can be derived:

$$\begin{aligned}
\mathbf{C}_{\mathbf{Z}} &= \mathrm{Cov}\left\{\left(\mathbf{AY}+\bar{\mathbf{A}}\bar{\mathbf{Y}}\right)\left(\mathbf{AY}+\bar{\mathbf{A}}\bar{\mathbf{Y}}\right)'\right\} \\
&= \mathrm{Cov}\left\{\left(\mathbf{AY}+\bar{\mathbf{A}}\bar{\mathbf{Y}}\right)\left(\left(\mathbf{AY}\right)'+\left(\bar{\mathbf{A}}\bar{\mathbf{Y}}\right)'\right)\right\} \\
&= \mathrm{Cov}\left\{\left(\mathbf{AY}+\bar{\mathbf{A}}\bar{\mathbf{Y}}\right)\left(\mathbf{Y}'\mathbf{A}'+\bar{\mathbf{Y}}'\bar{\mathbf{A}}'\right)\right\} \\
&= \mathrm{Cov}\left\{\mathbf{AYY}'\mathbf{A}'+\mathbf{AY}\bar{\mathbf{Y}}'\bar{\mathbf{A}}'+\bar{\mathbf{A}}\bar{\mathbf{Y}}\mathbf{Y}'\mathbf{A}'+\bar{\mathbf{A}}\bar{\mathbf{Y}}\bar{\mathbf{Y}}'\bar{\mathbf{A}}'\right\} \\
&= \mathbf{AC}_{\mathbf{Y}}\mathbf{A}'+\mathbf{AC}_{\mathbf{Y}\bar{\mathbf{Y}}}\bar{\mathbf{A}}'+\bar{\mathbf{A}}\mathbf{C}_{\bar{\mathbf{Y}}\mathbf{Y}}\mathbf{A}'+\bar{\mathbf{A}}\mathbf{C}_{\bar{\mathbf{Y}}}\bar{\mathbf{A}}'
\end{aligned}$$

Derivation of Covariance Integrals

The variance (1st covariance type) can be expressed in terms of the expected value of $\bar{Y}_i$ as:

$$\begin{aligned}
\mathrm{Cov}\{\bar{Y}_m(\mathbf{u}),\bar{Y}_m(\mathbf{u})\} &= \mathrm{Var}\{\bar{Y}_m(\mathbf{u})\} \\
&= E\left\{\bar{Y}_m(\mathbf{u})^2\right\}-E\left\{\bar{Y}_m(\mathbf{u})\right\}^2
\end{aligned}$$

The variable Y_i has an expected value (mean) of 0, so it follows that:

$$\begin{aligned}
E\left\{\bar{Y}_m(\mathbf{u})\right\} &= E\left\{\frac{1}{v}\int_v w_m(\mathbf{u})Y_m(\mathbf{u})d\mathbf{u}\right\} \\
&= \frac{1}{v}\int_v w_m(\mathbf{u})E\left\{Y_m(\mathbf{u})\right\}d\mathbf{u} = 0 \text{ as } E\left\{Y_m(\mathbf{u})\right\}=0
\end{aligned}$$

Also note that the variance of an underlying variable is:

$$\begin{aligned}
\mathrm{Cov}\{Y_m(\mathbf{u}),Y_m(\mathbf{u})\} &= E\left\{Y_m(\mathbf{u})^2\right\}-E\left\{Y_m(\mathbf{u})\right\}^2 \\
&= E\left\{Y_m(\mathbf{u})^2\right\} \text{ as } E\left\{Y_m(\mathbf{u})\right\}=0
\end{aligned}$$

Therefore, the variance of the weighted, volume averaged variable can be written in terms of the covariance function of the underlying Y_i variable:

$$\begin{aligned}
\mathrm{Cov}\{\bar{Y}_m(\mathbf{u}),\bar{Y}_m(\mathbf{u})\} &= E\left\{\bar{Y}_m(\mathbf{u})^2\right\} \\
&= E\left\{\left(\frac{1}{v}\int_v w_m(\mathbf{u})Y_m(\mathbf{u})d\mathbf{u}\right)^2\right\} \\
&= E\left\{\frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1)w_m(\mathbf{u}_2)Y_m(\mathbf{u}_1)Y_m(\mathbf{u}_2)d\mathbf{u}_1d\mathbf{u}_2\right\} \\
&= \frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1)w_m(\mathbf{u}_2)E\left\{Y_m(\mathbf{u}_1)Y_m(\mathbf{u}_2)\right\}d\mathbf{u}_1d\mathbf{u}_2 \\
&= \frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1)w_m(\mathbf{u}_2)\,\mathrm{Cov}\left\{Y_m(\mathbf{u}_1),Y_m(\mathbf{u}_2)\right\}d\mathbf{u}_1d\mathbf{u}_2 \\
&= \frac{1}{v^2}\int_v\int_v w_m(\mathbf{u}_1)w_m(\mathbf{u}_2)C_{Y_m}\left(\mathbf{u}_1-\mathbf{u}_2\right)d\mathbf{u}_1d\mathbf{u}_2
\end{aligned}$$